Zero-Temperature Attenuation and Transverse Spin Dynamics in Fermi Liquids. I. Generalized Landau Theory

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This is the first in a series of papers on a consistent microscopic theory of transverse dynamics in spin-polarized or binary Fermi liquids. We start from exact microscopic equations in Green's functions at zero temperatures and consider slightly inhomogeneous perturbations. The transverse dynamics is described by an integral equation in a 4D momentum space with inevitable spatial and temporal non-localities. This equation can be reduced only to a set of two coupled equations for partial transverse densities corresponding to independent contributions to a transverse magnetic moment from transverse components of slightly tilted up and down spins. It is shown that, in contrast to previous phenomenological theories of polarized Fermi liquids, these equations reduce to a single Landau-like kinetic equation only in cases of low polarization or density. This implies the existence of two different sorts of (attenuating) transverse quasi-particles. The molecular field (an analog of a Landau function) has a form of a 4-component non-local operator. This interaction operator is expressed via the off-diagonal component of the exact irreducible vertex with the help of some integral equation, and cannot be given, as it is usually assumed, as any limit of the full vertex. The proper Landau-like phenomenological approach corresponding to our exact microscopic equations, should operate with two types of attenuating transverse quasi-particles each oscillating between its Fermi surface and some other 3D surface in a 4D momentum space. The dephasing of inhomogeneous precession between two different types of dressed transverse quasi-particles leads to an inhomogeneous broadening which manifests itself as a peculiar zero-temperature relaxation.

1. INTRODUCTION

Most of the interesting macroscopic quantum phenomena in spin-polarized quantum systems manifest themselves in spin dynamics (see review 1). In equilibrium, both the Hamiltonian and the density matrix can

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be made diagonal in spins, and the particles are in pure spin-up and spin-down spin states. Near equilibrium, the spin dynamics can be separated in many cases into two major groups of longitudinal and transverse processes. The longitudinal processes cover the changes in the component of a magnetic moment parallel to equilibrium magnetization and/or external magnetic field, while the particles remain in pure spin states. The transverse processes, like spin waves, cover the dynamics of small transverse components of magnetization and involve off-diagonal (mixed) spin states of particles. Such a separation is possible when the particles' interaction is an exchange one and does not depend on spins of interacting particles, and when one is interested in states not very far from equilibrium.

The longitudinal phenomena include such transport processes as diffusion of longitudinal magnetization and longitudinal (dipole or spin-lattice) spin relaxation. The basics of longitudinal processes are quite clear (see, e.g., Ref. 1), though, of course, the details might be not at all trivial.

The situation with transverse spin dynamics is very different. Here it is often unclear even qualitatively what to expect in most general cases corresponding to dense highly polarized quantum systems especially at low (or zero) temperatures. What we know reduces to the following.

The transverse spin dynamics in spin-polarized quantum (Fermi) systems is dominated by a strong internal molecular field caused by a coherent part of particles' interaction. The general symmetry arguments[2] ensure that the equations of motion for transverse components of a macroscopic magnetic moment take the form of the Leggett equations[3] giving rise to a collective mode in the form of the well-known Silin spin waves with a quadratic dispersion law.[4] One does not have any problems at low spin polarizations when one can apply the standard Landau theory of Fermi liquids.[5] Then all relevant macroscopic characteristics are expressed through the scattering cross-section for quasi-particles and the harmonics of the usual Landau function describing the Fermi liquid interaction; the attentuation of the Silin waves[4] remains negligible at low temperatures as far as the polarization is low.

The problems arise at higher polarizations and are due to a not very clear character of transverse quasi-particles, their attenuation, and to non-local effects (a brief summary of the problems with highly polarized Fermi liquids can be found, e.g., in Ref. 6). Here the main questions concern the possibility of a consistent definition of transverse quasi-particles, and the magnitude of attenuation of the Silin waves at low temperatures when all standard relaxation processes are negligible. These questions are closely related to a more fundamental problem of the applicability and possible renormalizations for the Landau–Silin and Leggett–Rice theories for highly polarized Fermi liquids.

It has been known for some time,[7] that the transverse relaxation in spin dynamics of spin-polarized quantum systems differs considerably from the longitudinal one. The main difference is an irrelevance of standard conservation laws in case of transverse processes which may lead to a finite attenuation should not be a relaxation in a conventional sense, but rather an effective processes are already frozen. The origin of such a zero-temperature attenuation should not be a relaxation in a conventional sense, but rather a effective dephasing of oscillations in inhomogeneous situations resulting in some sort of inhomogeneous broadening. The problem becomes even more complicated if one takes into account a non-locality of interaction which contributes to the attenuation because of complex off-shell derivatives of the vertex function. As we will see, the zero-temperature attenuation is always associated with off-shell contributions even without any non-local effects. This problem, in turn, makes it very important to choose properly a set of dynamic variables for a Landau-like description of transverse phenomena, and to take into account important temporal and spatial non-localities.

The main goal of this series of papers is to develop a general consistent microscopic and semi-microscopic description of transverse phenomena in spin-polarized Fermi liquids. In the first paper we will derive the basic equations, and will try to get an insight into the physical nature of the zero-temperature attenuation in transverse dynamics. This will help us to formulate a generalized version of a phenomenological Landau-type theory applicable for highly polarized Fermi liquids. As we will see, standard versions of the Landau theory do not provide even a proper form of phenomenological equations for transverse dynamics in dense highly polarized Fermi liquids.

Simultaneously, we will get a concise understanding of the zero-temperature transverse attenuation. Lately Jeon and Mullin[8] have confirmed that the zero-temperature attenuation[7] does exist for polarized Fermi gases. However, the calculations were restricted to very dilute systems where the dissipation is small anyway, and were based on a Boltzmann-like kinetic equation whose applicability is not very clear at low temperatures. As we will see below, the proper kinetic equation has a different and a very unconventional form. Another concern is the use in Ref. 8 of the Born approximation for the Boltzmann equation: the straightforward application of the perturbation expansion[9] to transverse dynamics often leads to some anomalous contributions which might be cancelled out[10] when the results are expressed—as they should be—in terms of the exact vertex part (the T-matrix), and not the matrix elements of the interaction potential. What is more, the calculations[8] for extremely dilute gases do not take into account, quite rightfully, the non-local contributions which are very important for denser systems.[6] We will show below that proper transverse dynamics'

equations are highly non-local and reduce to a single kinetic equation in some "transverse density" only for vanishingly small densities of a Fermi system.

The paper is organized as follows. In the next section we derive the exact microscopic equations (including the non-locality) at zero temperature. Then, in Sec. 3, we will reduce these equations to a Landau-like semi-microscopic form and express a generalized non-local Landau interaction operator via an exact irreducible vertex. This, in turn, will allow us to get an exact spin-wave spectrum including the zero-temperature attenuation, and to formulate an adequate phenomenological approach (Sec. 4). In Sec. 4 we also discuss the shortcomings of the standard equations of the Landau theory for transverse spin dynamics and its possible generalizations. As we will see, the generalized equations should include time and spacial non-locality destroying a standard quasi-particle picture for transverse excitations. Instead, we arrive at a two-excitation description; the dephasing of inhomogeneous precession for these two types of excitations provides a clear reason for the zero-temperature attenuation. The last section contains a brief discussion and a summary. The following two parts of this series will contain the applications of the general results to dilute Fermi systems (Part II) and polarization/field expansions (Part III).

2. EXACT MICROSCOPIC EQUATIONS FOR TRANSVERSE SPIN DYNAMICS

In principle, what we would like to have is a generalized version of the Landau theory of Fermi liquids covering transverse phenomena and including retardation and spatial non-locality. In contrast to longitudinal phenomena for which such a generalization is rather straightforward (see, e.g., Refs. 1, 2, 11–17 and references therein), it is not obvious at all that one can develop a Landau-type theory for transverse processes corresponding to the dynamics of mixed spin states of the Landau quasi-particles. The difficulty here is associated with the mere principle of long-lived quasi-particles: it seems impossible to introduce off-diagonal (in spin space) single-particle states without a strong attenuation (see, e.g., Ref. 6). Therefore, we base our description on exact equations for real particles rather than try to introduce some quasi-particles. Only in the very end we will learn how and to what extent one can revitalize the Landau-like quasi-particle picture.

In this paper we deal with systems at zero temperature. On one hand it allows us to obtain the exact and easily understandable results, and on the other makes all the calculations easier. The generalization to higher temperatures will be given elsewhere.

At zero temperature we can start from the standard Dyson equation rather than from Keldysh or Kadanoff–Baym diagrammatic techniques for advanced and retarded functions:

$$G_{12} = G_{12}^{(0)} + G_{14}^{(0)} \Sigma_{43} G_{32},$$ (1)

$$G_{12} = G_{12}^{(0)} + G_{14} \Sigma_{43} G_{32}^{(0)},$$ (2)

and we will need both of these seemingly equivalent equations. Here G and $G^{(0)}$ are the Green's functions for interacting (dressed) and non-interacting (bare) fermions, and Σ is the self-energy part. Each of the indices represents a complete set of variables for a particle including the coordinates (or momenta), frequencies, energies, spin states, etc.

Applying the operators $\hat{G}_1^{(0)-1}$ to the first and $\hat{G}_2^{(0)*-1}$ to the second of the equations (1), and subtracting the results from each other, we get:

$$(G_2^{(0)*-1} - G_1^{(0)-1}) G_{12} = -\Sigma_{13} G_{32} + G_{13} \Sigma_{32}$$ (3)

where

$$\hat{G}_1^{(0)-1} = i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} + \mu + \beta \hat{\sigma}^z H,$$

and the same for $\hat{G}_2^{(0)}$.

In order to examine the transverse spin dynamics, we make a small variation of G and a related variation in Σ:

$$G' = G + \delta G, \qquad \Sigma' = \Sigma + \delta \Sigma$$

The corresponding variation of Eq. (3) should be calculated using the identity expressing small variations of the mass operator through the irreducible vertex function in the particle-hole channel $\tilde{\Gamma}$[18] (see Appendix A):

$$\delta \Sigma_{12} = -i \tilde{\Gamma}_{11',22'} \delta G_{1'2'}$$ (4)

As a result, the linearized variation of Eq. (3) takes the form

$$(G_2^{(0)*-1} - G_1^{(0)-1}) \delta G_{12}$$

$$= -\Sigma_{13} \delta G_{32} + \delta G_{13} \Sigma_{32} + i \tilde{\Gamma}_{11',33'} \delta G_{3'1'} G_{32} - i G_{13} \tilde{\Gamma}_{33',22'} \delta G_{2'3'}$$ (5)

All functions, G, Σ, and $\tilde{\Gamma}$, are 2×2 or ($\tilde{\Gamma}$) 4×4 matrices in spin space. The diagonal components of G and Σ correspond to pure spin-up ($\uparrow$) and spin-down ($\downarrow$) states. Only these components enter the unperturbed (equilibrium) values of G and Σ in Eq. (5), while the perturbation δG is allowed to have all four spin components. The spin matrix of the vertex part $\tilde{\Gamma}$ corresponds to the spin-conserving exchange interaction; as a result, $\tilde{\Gamma}_{\alpha\beta,\gamma\delta}$ (all spin indices have the values $\uparrow$ or $\downarrow$) has only the components for which the numbers of $\uparrow$ (and $\downarrow$) before and after the comma are equal to each other.

We are interested in transverse phenomena which are described by the off-diagonal components of Eq. (5), i.e. the equation in $\delta G_{\uparrow\downarrow}$. We will use the mixed representation

$$\delta G_{\uparrow\downarrow}(p_0, \mathbf{p}; t, \mathbf{r}) = g(p_0, \mathbf{p}) \exp(-i\omega t + i\mathbf{k}\mathbf{r}), \tag{6}$$

in which the off-diagonal component of Eq. (5) reduces to an integral equation

$$\left(\delta\omega - \frac{\mathbf{p}\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)\right) g(P)$$

$$= -i \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')\left[G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)\right] g(P') \frac{d^4 P'}{(2\pi)^4} \tag{7}$$

where p_0 and $\mathbf{p}$ are the temporal (frequency) and spatial components of the 4-vector P, and ω and $\mathbf{k}$ are the components of the 4-vector K, while $\delta\omega = \omega - \Omega_0$, $\Omega_0 = 2\beta H$ is the Larmor frequency for bare fermions, β is the magnetic moment of the fermions, $\mathbf{H}$ is the external magnetic field (here and below we assume $\hbar = 1$).

We will consider relatively smooth inhomogeneities corresponding to small $\mathbf{k}$. On one hand, only this case conforms with standard experimental setups, and, on the other, allows a reasonable comparison with a Landau-like approach and macroscopic Leggett equations. The smallness of $\mathbf{k}$ will also mean that $\delta\omega$ should be small. However, this does not mean that the frequency ω itself is small: the Larmor frequency Ω_0 and, therefore, the frequency ω remain arbitrary. What is more, the smallness of $\delta\omega$ does not even mean that we are considering a "hydrodynamic" regime $\delta\omega\tau_\perp \ll 1$: our results are applicable in a collisionless Silin regime $\delta\omega\tau_\perp \gg 1$ as well (see Ref. 1; $\tau_\perp$ is the transverse relaxation time).

Another very important feature of Eq. (7) is that it contains the Green's functions and mass operators only for pure spin states, and does not demand any preliminary information on the properties of the mixed states.

Note, that Eq. (7) is similar to the equation for the pole of the mixed component of the two-particle Green's function (see Ref. 19).

The eigennumbers of the integral equation (7) in $g(P)$ will give us the spectrum of spin waves (Silin type waves) $\delta\omega(\mathbf{k}) \propto k^2$. In case of low polarizations, the Fermi momenta for spin-up and spin-down particles are close to each other, $p_\uparrow \approx p_\downarrow \approx p_F$, the singular part of the difference $G_{\uparrow\uparrow}(\mathbf{p}) - G_{\downarrow\downarrow}(\mathbf{p})$ is proportional to $\delta(p - p_F)$, and the spectrum is real and reduces to the well-known Silin–Leggett equations[3,4] (see Sec. 3 and the third paper of this series for more details).

In case of a homogeneous spin precession, $\mathbf{k} = 0$, the interaction should not renormalize the Larmor frequency, $\delta\omega = 0$, $\omega = \Omega_0$. This can be proved,

as it is usually done in similar situations in the standard theory of Fermi liquids,[20] with the help of small rotation of the magnetic field or the polarization axis. As a result, we get two very convenient expressions:

$$\Sigma_{\downarrow\downarrow}(p_0 + \beta H, \mathbf{p}) - \Sigma_{\uparrow\uparrow}(p_0 - \beta H, \mathbf{p})$$

$$= i \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(2\beta H; P, P')[G_{\downarrow\downarrow}(p_0' + \beta H, \mathbf{p}') - G_{\uparrow\uparrow}(p_0' - \beta H, \mathbf{p}')] \frac{d^4P'}{(2\pi)^4} \quad (8)$$

and

$$(\Sigma_{\downarrow\downarrow}(p_0 + \beta H, \mathbf{p}) - \Sigma_{\uparrow\uparrow}(p_0 - \beta H, \mathbf{p}))g_0(P)$$

$$= i \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(2\beta H, 0; P, P')$$

$$\times [G_{\downarrow\downarrow}(p_0 + \beta H, \mathbf{p}) - G_{\uparrow\uparrow}(p_0 - \beta H, \mathbf{p})]g_0(P') \frac{d^4P'}{(2\pi)^4} \quad (9)$$

The equation (9) is valid if

$$g_0(P) = \text{const} \times \left[G_{\downarrow\downarrow}\left(p_0 + \frac{\Omega_0}{2}, \mathbf{p}\right) - G_{\uparrow\uparrow}\left(p_0 - \frac{\Omega_0}{2}, \mathbf{p}\right) \right] \quad (10)$$

and is, basically, an equivalent of Eq. (7).

We are interested in the case of small, but non-zero, inhomogeneities described by the lowest orders in the vector **k**. Then we can write the expansion for the function $g(P)$ as

$$g(p_0, \mathbf{p}) = g_0(p_0, p) + g_1(p_0, p)\mathbf{pk}$$

$$+ g_2(p_0, p)(\mathbf{pk})^2 + g_3(p_0, p)p^2k^2 + \cdots \quad (11)$$

while the frequency contains only the even powers of **k**:

$$\omega = \Omega_0 + \alpha k^2 + \cdots \quad (12)$$

In this paper we are interested in the main term in the spin wave spectrum $\alpha = \alpha' + i\alpha''$. On one hand, this coefficient contains all important information on the waves propagation (α') and the zero-temperature attenuation (α''). On the other hand, this accuracy is necessary and sufficient to recover the equations of spin dynamics in both hydrodynamic Leggett and high-frequency Silin limits (the iterrelations between hydrodynamic and collision limits for this type of problem with large internal molecular field frequencies are discussed in some detail in review).[1] Therefore we can truncate the expansion (11) immediately after the terms with k^2.

The value of α can be determined by substituting expansion (11) up to the terms k^2 into Eq. (7). In doing so, it is convenient to introduce the

arguments for the irreducible vertex part $\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')$ in such a way as to ensure the zero value of the derivative $\partial\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')/\partial k = 0$ (see Appendix B). As usual in such problems, it is sufficient to restrict oneself only to the first two terms in the expansion for $g(P)$, $g(P) = g_0(p_0, p) + g_1(p_0, p)$**pk**: if one wants to get the eigenvalues of a non-degenerate operator in the second order, it is sufficient to know the eigenfunctions just in the first order.

Note, that the k^2-terms enter the equations not only because of an explicit dependence on **k** in the bracket in the l.h.s. of Eq. (7) and the presence of **k** in the spatial components of the 4-vector K, but also because of the presence of the frequency $\delta\omega \propto k^2$ in the temporal component of the 4-vector $K = (\Omega_0/2 + \delta\omega, \mathbf{k})$ in the arguments of the Green's functions, mass operators and the vertex in Eq. (7). This leads to a presence of a denominator S in the following expression for the coefficient α:

$$
\begin{aligned}
\alpha = \frac{1}{S}\Bigg\{ & \frac{1}{4}\int \left(\Sigma_{\downarrow\downarrow P_i P_j}\left(P + \frac{K_0}{2}\right) - \Sigma_{\uparrow\uparrow P_i P_j}\left(P - \frac{K_0}{2}\right)\right)\frac{k_i k_j}{k^2}\, g_0(P)\, d^4 P \\
& - i\int\int \left[\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow k_i k_j}(K_0; P, P')\left(G_{\downarrow\downarrow}\left(P + \frac{K_0}{2}\right) - G_{\uparrow\uparrow}\left(P - \frac{K_0}{2}\right)\right)\right. \\
& \left. + \frac{1}{4}\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K_0; P, P')\left(G_{\downarrow\downarrow P_i P_j}\left(P + \frac{K_0}{2}\right) - G_{\uparrow\uparrow P_i P_j}\left(P - \frac{K_0}{2}\right)\right)\right] \\
& \times \frac{k_i k_j}{k^2}\, g_0(P')\, d^4 P\, d^4 P'/(2\pi)^4 \\
& + \int d^4 P \frac{(\mathbf{pk})^2}{k^2}\, g_1(P)\left[\frac{1}{m} + \frac{1}{2}\left(\Sigma_{\downarrow\downarrow P}\left(P + \frac{K_0}{2}\right) + \Sigma_{\uparrow\uparrow P}\left(P - \frac{K_0}{2}\right)\right)\right. \\
& \left. - \frac{i}{2}\int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(k_0, 0; P', P)\left(G_{\downarrow\downarrow P'}\left(P' + \frac{K_0}{2}\right) + G_{\uparrow\uparrow P'}\left(P' - \frac{K_0}{2}\right)\right)\frac{d^4 P'}{(2\pi)^4}\right]\Bigg\}
\end{aligned}
\tag{13}
$$

where the denominator S itself is equal to the integral

$$
\begin{aligned}
S = \int \Bigg\{ & \left[1 - \frac{1}{2}\left(\Sigma_{\downarrow\downarrow P_0}\left(P + \frac{K_0}{2}\right) + \Sigma_{\uparrow\uparrow P_0}\left(P - \frac{K_0}{2}\right)\right)\right] \\
& + i\int \left[\tilde{\Gamma}_{k_0}(K_0; P, P')\left(G_{\downarrow\downarrow}\left(P' + \frac{K_0}{2}\right)\right.\right. \\
& \left. - G_{\uparrow\uparrow}\left(P' - \frac{K_0}{2}\right)\right) + \frac{1}{2}\tilde{\Gamma}(K_0; P, P')\left(G_{\downarrow\downarrow P_0}\left(P' + \frac{K_0}{2}\right)\right. \\
& \left.\left. + G_{\uparrow\uparrow P_0}\left(P' - \frac{K_0}{2}\right)\right)\right]\frac{d^4 P'}{(2\pi)^4}\Bigg\}\left[G_{\downarrow\downarrow}\left(P + \frac{K_0}{2}\right) - G_{\uparrow\uparrow}\left(P - \frac{K_0}{2}\right)\right] d^4 P
\end{aligned}
\tag{14}
$$

while the 4-vector $K_0 = (\Omega_0, \mathbf{0})$, $\Omega_0 = 2\beta H$, arrows stand for spin-up and spin-down states,

$$\mathbf{p}\Sigma_p = \frac{\partial \Sigma}{\partial \mathbf{p}}, \qquad \Sigma_{p_i p_j} = \frac{\partial^2 \Sigma}{\partial p_i \partial p_j},$$

$$\mathbf{p}G_p = \frac{\partial G}{\partial \mathbf{p}}, \qquad G_{p_i p_j} = \frac{\partial^2 G}{\partial p_i \partial p_j}, \qquad \tilde{\Gamma}_{k_i k_j} = \frac{\partial^2 \tilde{\Gamma}}{\partial k_i \partial k_j} \tag{15}$$

and the main term in expansion (11), $g_0(P)$, is given by Eq. (10).

We still have to express g_1 in Eq. (13) through g_0 (10) with the help of Eq. (7) which leads us to the integral equation

$$\left[\Sigma_{\uparrow\uparrow}\left(P - \frac{K_0}{2}\right) - \Sigma_{\downarrow\downarrow}\left(P + \frac{K_0}{2}\right)\right]g_1(P) - i\int \frac{\mathbf{p'k}}{\mathbf{pk}}\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K_0; P, P')$$

$$\times \left[G_{\downarrow\downarrow}\left(P + \frac{K_0}{2}\right) - G_{\uparrow\uparrow}\left(P - \frac{K_0}{2}\right)\right]g_1(P')\frac{d^4P'}{(2\pi)^4}$$

$$= \left[\frac{1}{m} + \Sigma_{\downarrow\downarrow p}\left(P + \frac{K_0}{2}\right) + \Sigma_{\uparrow\uparrow p}\left(P - \frac{K_0}{2}\right)\right]g_0(P)$$

$$-\frac{i}{2}\int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K_0; P, P')\left(G_{\downarrow\downarrow p}\left(P + \frac{K_0}{2}\right)\right.$$

$$\left. + G_{\uparrow\uparrow p}\left(P - \frac{K_0}{2}\right)\right)g_0(P')\frac{d^4P'}{(2\pi)^4} \tag{16}$$

The set of Eqs. (13)–(16) determines the spectrum of collective transverse excitations including their zero-temperature attenuation. It is possible to simplify Eq. (13) using relations (8), (9), and, after some algebra, to reduce the expression for α to

$$\alpha = -\frac{i}{S}\left\{\frac{k_i k_j}{k^2}\int\int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow k_i k_j}(K_0; P, P')\left(G_{\downarrow\downarrow}\left(P + \frac{K_0}{2}\right)\right.\right.$$

$$\left. - G_{\uparrow\uparrow}\left(P - \frac{K_0}{2}\right)\right)\left(G_{\downarrow\downarrow}\left(P' + \frac{K_0}{2}\right)\right.$$

$$\left. - G_{\uparrow\uparrow}\left(P' - \frac{K_0}{2}\right)\right)\frac{d^4P d^4P'}{(2\pi)^8} + i\int \frac{d^4P}{(2\pi)^4}\frac{(\mathbf{pk})^2}{k^2}g_1(P)$$

$$\times\left[\frac{1}{m} + \frac{1}{2}\left(\Sigma_{\downarrow\downarrow p}\left(P + \frac{K_0}{2}\right) + \Sigma_{\uparrow\uparrow p}\left(P - \frac{K_0}{2}\right)\right)\right.$$

$$-\frac{i}{2}\int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K_0; P', P)\left(G_{\downarrow\downarrow p'}\left(P' + \frac{K_0}{2}\right)\right.$$

$$\left.\left. + G_{\uparrow\uparrow p'}\left(P' - \frac{K_0}{2}\right)\right)\right]\frac{d^4P'}{(2\pi)^4}\right\} \tag{17}$$

where the 4-vector $K_0 = (\Omega_0, 0)$ contains only the Larmor frequency $\Omega_0 = 2\beta H$.

Of course, these equations cannot be solved exactly for a Fermi liquid of a general type for which we know neither the kernel of the integral equations (16)–(17) (the vertex part), nor the exact expressions for the mass operators $\Sigma_{\downarrow\downarrow}, \Sigma_{\uparrow\uparrow}$ and Green's functions $G_{\downarrow\downarrow}, G_{\uparrow\uparrow}$. However, even at this stage we can already indicate the sources of imaginary contributions to the spin-wave spectrum α (17). Basically, there are two major sources for α'': the imaginary terms in the vertex $\tilde{\Gamma}$ and its derivatives, and large imaginary terms in the mass operators $\Sigma_{\uparrow\uparrow}, \Sigma_{\downarrow\downarrow}$ and the Green's functions $G_{\uparrow\uparrow}, G_{\downarrow\downarrow}$ at frequencies far away from the Fermi energies. The former terms contribute to the zero-temperature attenuation mostly due to a non-locality of the interaction and to imaginary contributions to the scattering amplitude in off-shell directions. The latter terms lead to the attenuation because single-particle states in Fermi liquids (quasi-particles) do not attenuate only very close to the Fermi surface. At high polarizations (i.e. if the difference in Fermi momenta for up and down spins $p_\uparrow - p_\downarrow$ is large) the differences $\Sigma_{\uparrow\uparrow} - \Sigma_{\downarrow\downarrow}$ and $G_{\uparrow\uparrow} - G_{\downarrow\downarrow}$ are non-zero everywhere between the Fermi surfaces. Thus these differences involve states which are far away from the corresponding Fermi surfaces, and, therefore, have large imaginary parts. Since these two sources of the zero-temperature attenuation have a somewhat different nature, the corresponding contributions manifest themselves under different circumstances. Later on, we will evaluate the zero-temperature attenuation of spin waves in different types of polarized Fermi liquids.

3. GENERALIZED THEORY OF FERMI LIQUIDS FOR TRANSVERSE PHENOMENA AT HIGH POLARIZATIONS

In principle, the above results are sufficient to determine the transverse spectrum and the zero-temperature relaxation in different systems. However, it is desirable to put the same equations into the context of a generalized Landau theory of Fermi liquids. On one hand, it will allow an easy comparison with the well-known Silin and Leggett–Rice results at low polarizations and will ensure a good reference point. And, on the other hand, we will be able to understand all the shortcomings of the standard Landau theory with respect to transverse phenomena at high polarizations. What is more, as an important by-product, we will eliminate the singularities in the above integral equations. In doing so, we will see that the standard quasi-particle description fails and becomes absolutely unapplicable. The whole macroscopic picture should be changed starting from the main assumptions. The main change concerns the incorporation of different temporal and

spatial non-localities as a result of which the equations in Green's functions resist the attempts to rewrite them as a single kinetic (transport) equation for an off-diagonal (mixed in spins) component of some single-particle (quasi-particle) distribution. The only thing one can do is to reduce these equations in Green's functions to a set of two coupled equations in some attenuating partial "transverse densities". As a result, the transverse Landau interaction assumes a form of a peculiar 4-component matrix operator. Needless to say, this major change occurs only for transverse dynamics of highly polarized systems; the equations of longitudinal dynamics can still be written in the form of a slightly modified Landau kinetic equation for semi-classical distribution functions of up and down spins.

From the technical standpoint, we will deal with and benefit from the singularities in the kernels of the above integral equations. To the best of our knowledge, these singularities were not approached consistently despite numerous previous attempts to develop a microscopic theory of transverse phenomena (some of the recent results are described in Ref. 16).

The success of the Landau theory is based on the exploitation of the singularity in the particle-hole channel with a small momentum transfer which corresponds to the singular part of the product of the Green's functions $G_{\downarrow\downarrow}(P + K/2)G_{\uparrow\uparrow}(P - K/2)$. The main single assumption of this paper is that the Green's functions for particles *in pure spin states* have a (single) singularity (a pole) on their respective Fermi surfaces; we do not make any assumptions about analytical properties of mixed states. Of course, such an assumption reduces the applicability of results only to conventional normal Fermi liquids, and leaves aside more exotic systems like marginal Fermi liquids.[21] In case of transverse dynamics, this singularity is crucial for the integrand in Eq. (7) since, by definition,

$$G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)$$

$$= \left[\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)\right]G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right)G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right) \quad (18)$$

The product of the Green's functions (18) can be split into a singular and regular parts ϕ_r and ϕ_s, which may be written in the form

$$G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right)G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)$$

$$= i\phi_r\left(P + \frac{K}{2}, P - \frac{K}{2}\right) + \frac{i\phi_s\left(P + \frac{K}{2}, P - \frac{K}{2}\right)}{\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)} \quad (19)$$

The singular part originates from poles of the Green's functions for spin-up and spin-down particles, and is due to the pinching of the contour of integration over p_0 between those poles of $G_{\downarrow\downarrow}(P+K/2)$ and $G_{\uparrow\uparrow}(P-K/2)$ which are located on the opposite sides of the line $\text{Im } p_0 = 0$. If we are interested in the residue of the integrand in the pole corresponding to one of these Green's functions, the denominator of the other will be exactly equal to the term in the square brackets in Eq. (18) and will cancel it:

$$
G_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P-\frac{K}{2}\right)
$$

$$
= i\phi_r\left(P+\frac{K}{2}, P-\frac{K}{2}\right)\left[\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P-\frac{K}{2}\right)\right]
$$

$$
+ i\phi_s\left(P+\frac{K}{2}, P-\frac{K}{2}\right) \tag{20}
$$

Now one has to substitute Eqs. (18)–(20) into the integral equation (7), and to regroup regular and singular terms. After some algebra (see Appendix C), this equation reduces to the integral equation with a regular kernel,

$$
\left(\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P-\frac{K}{2}\right)\right)\tilde{g}(P)
$$

$$
= \int \mathfrak{C}(K; P, P')\phi_s\left(P'+\frac{K}{2}, P'-\frac{K}{2}\right)\tilde{g}(P')\frac{d^4P'}{(2\pi)^4} \tag{21}
$$

where we have used the notation

$$
g(P) = \left[G_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P-\frac{K}{2}\right)\right]\tilde{g}(P) \tag{22}
$$

and have introduced the regular function $\mathfrak{C}(K; P, P')$ in ther kernel with the help of the integral equation

$$
\mathfrak{C}(K; P, P') = \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P') + \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, Q)
$$

$$
\times \phi_r(Q+K/2, Q-K/2)\mathfrak{C}(K; Q, P')\frac{d^4Q}{(2\pi)^4} \tag{23}
$$

The r.h.s. of Eq. (22) contains explicitly the difference of two Green's functions. As a result, the function $\tilde{g}(P)$ is not singular, while $g(P)$ is proportional to the same difference, and, of course, contains a singularity.

The function $\mathfrak{C}(K; P, P')$ (23) will be used to generate—though in a non-trivial way—the generalized Landau interaction for transverse

phenomena in polarized Fermi liquids. It serves for highly polarized systems as an analog of the antisymmetric (in spins) part F^a of the Landau function Γ^ω which is usually defined for non-polarized systems as the following limit of a full vertex[5,18,22]:

$$\frac{2\pi^2}{Z^2 p_F m^*} F^a(\mathbf{p}_F, \mathbf{p}_F') \equiv \Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow}(\mathbf{p}_F, \mathbf{p}_F')$$

$$= \lim_{\omega \to 0} \lim_{k \to 0} \Gamma_{\uparrow\downarrow,\downarrow\uparrow}(\omega, \mathbf{k}; 0, \mathbf{p}_F; 0', \mathbf{p}_F') \qquad (24)$$

However, it is possible to give a different, but a completely equivalent definition for Γ^ω (see Refs. 18, 22 and Appendix C):

$$\Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow}(P, P') = \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(P, P')$$

$$+ \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(P, Q)\phi_r(Q, Q)\Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow}(Q, P')\frac{d^4Q}{(2\pi)^4} \qquad (25)$$

or, in the operator form,

$$\Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow} = (1 - \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}\phi_r)^{-1}\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow} \qquad (26)$$

For longitudinal processes the equivalence of equations analogous to (24) and (25) holds even for highly polarized systems. It is not so for transverse processes, though some unsuccessful attempts to introduce the Landau interaction function as a limit of the full vertex part are also made from time to time for highly polarized Fermi liquids (e.g. Ref. 16 and references therein).

At high polarizations, and this is a very important point, the Landau interaction function for transverse phenomena cannot be introduced as some limit of a full vertex by modifying Eq. (24). The reason is the inevitability and unsuppressability of the singularity ϕ_s in the full vertex Γ in polarized systems. In non-polarized systems, this singularity can be marginally suppressed in the limit $\mathbf{k} \to 0$. In this limit both Fermi spheres, as well as the poles of the Green's functions, merge. This eliminates the singularity which is proportional to the volume in the $\mathbf{p}$-space in which the poles of the Green's functions for spin-ups and spin-downs are on the different sides of the contour of integration over p_0 (see Appendix D). In polarized Fermi liquids, the radii of the Fermi spheres for spin-ups and spin-downs are different, and the poles of the Green's functions for spin-ups and spin-downs cannot merge completely, remaining on the different sides of the integration contour within the finite volume in the $\mathbf{p}$-space. Therefore, any limit of the full vertex function $\Gamma_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')$ contains the above singularity, while the irreducible vertex function $\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')$, and, consequently, the generalized Landau interaction $\mathfrak{S}(K; P, P')$, defined by Eq. (23), do not, and are regular.

Note, that the formal operator equation (26) remains valid at all polarizations. In essence, our integral equation (23) is a generalization of Eq. (25) for high polarizations, and includes the non-locality (finite $\delta\omega$ and $\mathbf{k}$) which is very important, as we will see later, at high polarizations.

So far, the functions ϕ_s and ϕ_r have not been uniquely and explicitly defined. Of course, there is always some ambiguity in separating singular and regular parts of the product of the Green's functions. Such a separation is based on the fact that the Green's function $G(p_0, \mathbf{p})$ should have a pole at $p_0 = \varepsilon(\mathbf{p})$ and, therefore can be presented as

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = \frac{Z_{\uparrow,\downarrow}(p_0, \mathbf{p})}{p_0 - \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) + \mu + i0 \, \text{sign} \, p_0} + G^r_{\uparrow,\downarrow}(P) \tag{27}$$

where $Z_{\uparrow,\downarrow}(P)$ and $G^r_{\uparrow,\downarrow}(P)$ are some regular functions. These functions are not defined uniquely—certain changes in the numerator Z (proportional to the denominator) can be compensated by corresponding changes in G^r. In order to fix the definition of $Z_{\uparrow,\downarrow}(P)$ we will define it exactly as a residue of the Green's function in the pole $p_0 = \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) - \mu$, namely as a function $Z_{\uparrow,\downarrow}(\mathbf{p}) = Z_{\uparrow,\downarrow}(p_0 = \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) - \mu, \mathbf{p})$ introduced as

$$Z_{\uparrow,\downarrow}(\mathbf{p}) = \text{res} \, G_{\uparrow\uparrow,\downarrow\downarrow}(p_0, \mathbf{p})$$

which is equivalent to

$$Z_{\uparrow,\downarrow}^{-1}(\mathbf{p}) = \frac{\partial G_{\uparrow\uparrow,\downarrow\downarrow}^{-1}(p_0 = \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) - \mu, \mathbf{p})}{\partial p_0} \tag{28}$$

According to the definition of the mass operator,

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = [p_0 - p^2/2m \pm \beta H - \Sigma_{\uparrow\uparrow,\downarrow\downarrow}(P) - \mu + i0 \, \text{sign} \, p_0]^{-1},$$

Eq. (28) can be rewritten as

$$1 - \frac{1}{Z_{\uparrow,\downarrow}(\mathbf{p})} = \frac{\partial \Sigma_{\uparrow\uparrow,\downarrow\downarrow}(p_0 = \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) - \mu, \mathbf{p})}{\partial p_0} \tag{29}$$

In the absence of the field/polarization, one usually encounters only the constant $Z = Z(p_F)$ (cf. Eq. (24)). Now we can define the function ϕ_s (19), (20) (and, therefore, ϕ_r) as an unambiguous combination of δ- and θ-functions (see Appendix E):

$$\phi_s\left(P + \frac{K}{2}, P - \frac{K}{2}\right) = \pi\left[\theta\left(\varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) - \mu\right) - \theta\left(\varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) - \mu\right)\right]$$

$$\times \left[Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)\delta\left(p_0 + \frac{\delta\omega}{2} + \beta H - \varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) + \mu\right)\right.$$

$$\left. + \left(Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right)\delta\left(p_0 - \frac{\delta\omega}{2} - \beta H - \varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) + \mu\right)\right]$$

$$\tag{30}$$

Not surprisingly, the dependence of ϕ_s on p_0 is fairly simple and reduces to two δ-functions. However, as a result of a temporal non-locality, both δ-functions are centered at high polarizations, around two separate points with increasing distance between them with growing polarizations. As we will see shortly, the presence of these two separate δ-functions in Eq. (30) causes major complications and changes with respect to non-polarized or weakly polarized systems.

Eqs. (21), (23) and (30) form a complete set of equations necessary to describe the transverse dynamics. Since all the kernels of integral equations are now regular and unambiguously defined, the description is complete and well-defined.

In case of low polarizations one can simplify considerably the set of analogous equations by rewriting it as much simpler equations in quasi-particle (spin) densities thus reducing the dimensionality of all integral equations from 4-vectors P to usual 3D momenta $\mathbf{p}$. It is highly desirable to do something like that at arbitrary polarizations. Now we have two different δ-functions (30) containing p_0 which should be substituted into Eq. (21). These δ-functions trivialize integration over p_0' and reduce Eq. (21) to

$$\left(\delta\omega - \frac{\mathbf{p}\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}(P+K/2) + \Sigma_{\uparrow\uparrow}(P-K/2)\right)\tilde{g}(P)$$

$$= \frac{1}{2}\int\left[\mathfrak{C}(K; P, P_\downarrow')Z_\downarrow\left(\mathbf{p}'+\frac{\mathbf{k}}{2}\right)\tilde{g}(P_\downarrow') + \mathfrak{C}(K; P, P_\uparrow')Z_\uparrow\left(\mathbf{p}'-\frac{\mathbf{k}}{2}\right)\tilde{g}(P_\uparrow')\right]$$

$$\times\left[\theta_\downarrow(\mathbf{p}') - \theta_\uparrow(\mathbf{p}')\right]\frac{d^3p'}{(2\pi)^3} \tag{31}$$

where

$$K = (\delta\omega, \mathbf{k}), \qquad P_\downarrow = \left(\varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) - \mu - \frac{\delta\omega}{2} - \beta H, \mathbf{p}\right),$$

$$P_\uparrow = \left(\varepsilon_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right) - \mu + \frac{\delta\omega}{2} + \beta H, \mathbf{p}\right),$$

$$\delta\omega = \omega - 2\beta H, \qquad \theta_\downarrow(\mathbf{p}) = \theta\left(\varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) - \mu\right),$$

$$\theta_\uparrow(\mathbf{p}) = \theta\left(\varepsilon_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right) - \mu\right). \tag{32}$$

Generally, we are not interested in the eigenfunction $\tilde{g}(P)$ at arbitrary values of P. We want to know only its values on the 3D surfaces $P = P_{\uparrow,\downarrow}$ in the 4D space of momenta P. Therefore, we need to find only two functions

$\tilde{g}_\uparrow(\mathbf{p}) \equiv \tilde{g}(P_\uparrow)$ and $\tilde{g}_\downarrow(\mathbf{p}) \equiv \tilde{g}(P_\downarrow)$ in a 3D $\mathbf{p}$-space for which we can introduce the "transverse densities" $\delta n_\uparrow(\mathbf{p})$ and $\delta n_\downarrow(\mathbf{p})$ as

$$
\begin{aligned}
\delta n_\downarrow(\mathbf{p}) &= \tilde{g}_\downarrow(\mathbf{p}) \equiv \tilde{g}(P_\downarrow), \\
\delta n_\uparrow(\mathbf{p}) &= \tilde{g}_\uparrow(\mathbf{p}) \equiv \tilde{g}(P_\uparrow).
\end{aligned}
\tag{33}
$$

Then Eq. (31) reduces to a set of two coupled equations in transverse densities:

$$
W_\downarrow(K; \mathbf{p})\delta n_\downarrow(\mathbf{p}) = \frac{1}{2}\int [F_{\downarrow\downarrow}(K; \mathbf{p}, \mathbf{p}')\delta n_\downarrow(\mathbf{p}') + F_{\downarrow\uparrow}(K; \mathbf{p}, \mathbf{p}')\delta n_\uparrow(\mathbf{p}')]
$$

$$
\times (\theta_\downarrow(\mathbf{p}') - \theta_\uparrow(\mathbf{p}'))\frac{d^3 p'}{(2\pi)^3}
$$

$$
W_\uparrow(K; \mathbf{p})\delta n_\uparrow(\mathbf{p}) = \frac{1}{2}\int [F_{\uparrow\downarrow}(K; \mathbf{p}, \mathbf{p}')\delta n_\downarrow(\mathbf{p}') + F_{\uparrow\uparrow}(K; \mathbf{p}, \mathbf{p}')\delta n_\uparrow(\mathbf{p}')]
$$

$$
\times (\theta_\downarrow(\mathbf{p}') - \theta_\uparrow(\mathbf{p}'))\frac{d^3 p'}{(2\pi)^3}
\tag{34}
$$

with the following functions W:

$$
\begin{aligned}
W_\downarrow(K; \mathbf{p}) &= \left[\delta\omega - \frac{\mathbf{p}\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}\left(P_\downarrow + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P_\downarrow - \frac{K}{2}\right)\right]Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right), \\
W_\uparrow(K; \mathbf{p}) &= \left[\delta\omega - \frac{\mathbf{p}\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}\left(P_\uparrow + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P_\uparrow - \frac{K}{2}\right)\right]Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right),
\end{aligned}
\tag{35}
$$

and the 2×2 interaction operator (matrix) $\hat{F}(K; \mathbf{p}, \mathbf{p}')$ which plays the role of the set of four generalized non-local Landau functions:

$$
F_{\downarrow\downarrow}(K; \mathbf{p}, \mathbf{p}') = Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)Z_\downarrow\left(\mathbf{p}' + \frac{\mathbf{k}}{2}\right)\mathfrak{C}(K; P_\downarrow, P'_\downarrow),
$$

$$
F_{\downarrow\uparrow}(K; \mathbf{p}, \mathbf{p}') = Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)Z_\uparrow\left(\mathbf{p}' - \frac{\mathbf{k}}{2}\right)\mathfrak{C}(K; P_\downarrow, P'_\uparrow)
$$

$$
F_{\uparrow\downarrow}(K; \mathbf{p}, \mathbf{p}') = Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right)Z_\downarrow\left(\mathbf{p}' + \frac{\mathbf{k}}{2}\right)\mathfrak{C}(K; P_\uparrow, P'_\downarrow),
$$

$$
F_{\uparrow\uparrow}(K; \mathbf{p}, \mathbf{p}') = Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right)Z_\uparrow\left(\mathbf{p}' - \frac{\mathbf{k}}{2}\right)\mathfrak{C}(K; P_\uparrow, P'_\uparrow)
\tag{36}
$$

Note, that the above functions $\delta n_\uparrow(\mathbf{p})$ and $\delta n_\downarrow(\mathbf{p})$ are not distribution functions of spin-up and spin-down quasi-particles. The functions $\delta n_\uparrow(\mathbf{p})$ and $\delta n_\downarrow(\mathbf{p})$ are rather partial contributions to a mixed spin component of

quasi-particles' density matrix from slightly tilted spin-ups and spin-downs (see below). Probably, a more appropriate, but much more cumbersome, notation should be $\delta n_\uparrow(\mathbf{p}) = \delta n_{\uparrow\downarrow}^{(\uparrow)}(\mathbf{p})$ and $\delta n_\downarrow(\mathbf{p}) = \delta n_{\uparrow\downarrow}^{(\downarrow)}(\mathbf{p})$.

The zeroth components of 4-vectors $P_{\uparrow,\downarrow}$ (32) are complex since the single particle spectra $\varepsilon_{\uparrow,\downarrow}(\mathbf{p})$ are complex functions of $\mathbf{p}$ with real values only on the corresponding Fermi surfaces. As a result, the above functions W and F, i.e. the coefficients in Eqs. (34) are complex except for the limits of vanishing polarizations, when the functions are taken only on the Fermi surfaces, or of vanishing densities, when the single-particle spectra are real everywhere. This explains, essentially, the presence of an imaginary part in α and the existence of a considerable zero-temperature attenuation (transverse relaxation $\tau_\perp$) at high polarizations/densities.

The split of the general equation into a set of two coupled equations in two densities $\delta n_{\downarrow,\uparrow}(\mathbf{p})$ is quite natural. The polarized system consists of two subsystems with spins up and down. When we slightly tilt the magnetization, particles with spins up and down start to process around the quantization axis. In homogeneous case, when the distributions are still the equilibrium ones (in a rotating frame), the precession frequencies are the same for both species with different spin projections and are equal to the bare Larmor frequency. However, if one introduces some deviation from equilibrium distributions, $\delta n_{\downarrow,\uparrow}(\mathbf{p})$, the precession frequencies become different because of the obvious asymmetry in the mean field Landau interaction with respect to spin projections. As a result, the equation splits into a system of two coupled equations (34). This spin-up–spin-down asymmetry is somewhat analogous to a usual particle-hole asymmetry for deressed particles, and is proportional to the degree of polarization. It disappears in cases of either weak polarizations or very dilute systems when the effective mean field interaction is constant. In both of these limiting cases the equations (34) merge into one equation (this will be disussed in more detail in Parts II and III of this series).

It is worth transforming Eqs. (34)–(36) in order to highlight this analogy with a two-component mixture of pure spin states. Later, this will also help us to establish a link with a possible phenomenological Landau-like description. By definition, the energy spectra of spin-up and spin-down particles, $\varepsilon_\uparrow(\mathbf{p})$ and $\varepsilon_\downarrow(\mathbf{p})$, are determined as the solutions of the following equations:

$$\varepsilon_\uparrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\uparrow\uparrow}(\varepsilon_\uparrow(\mathbf{p}) - \mu, \mathbf{p}) - \beta H,$$

$$\varepsilon_\downarrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\downarrow\downarrow}(\varepsilon_\downarrow(\mathbf{p}) - \mu, \mathbf{p}) + \beta H$$

$$(37)$$

We can also introduce spectra of some *pseudo*-particles' states defining

them by equations similar to Eqs. (37):

$$\tilde{\varepsilon}_\uparrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\uparrow\uparrow}(\varepsilon_\downarrow(\mathbf{p}+\mathbf{k}) - \mu - \delta\omega - 2\beta H, \mathbf{p}) - \beta H,$$

$$\tag{38}$$

$$\tilde{\varepsilon}_\downarrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\downarrow\downarrow}(\varepsilon(\mathbf{p}-\mathbf{k}) - \mu + \delta\omega + 2\beta H. \mathbf{p}) + \beta H$$

One gets into a *pseudo*-state with spin up, $\tilde{\varepsilon}_\uparrow(\mathbf{p}) \neq \varepsilon_\uparrow(\mathbf{p})$, by simply flipping the spin of a particle with spin down and nearly the same momentum ($\mathbf{k}$ is small), and shifting its energy by the *bare* Larmor frequency plus small $\delta\omega$. Analogously, one gets the spin-down *pseudo*-states, $\tilde{\varepsilon}_\downarrow(\mathbf{p}) \neq \varepsilon_\downarrow(\mathbf{p})$, by flipping the spin of a spin-up particle with a corresponding momentum and energy shifts. Such operations do not end up with real Fermi surface states because the difference between the Fermi energies of spin-ups and spin-downs (which is, essentially, the zero-temperature susceptibility of a highly polarized Fermi liquid),

$$\varepsilon_\downarrow(\mathbf{p}) - \varepsilon_\uparrow(\mathbf{p}) = 2\beta_1(\mathbf{p}, H)H, \tag{39}$$

is very different for dressed particles from the bare Larmor frequency $2\beta H$. Eq. (39) may be considered as a definition of the function $\beta_1(\mathbf{p}, H)$.

With the help of Eq. (39), Eqs. (38) can be rewritten as

$$\tilde{\varepsilon}_\uparrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\uparrow\uparrow}(\varepsilon_\uparrow(\mathbf{p}+\mathbf{k}) - \mu - \delta\omega + 2(\beta_1(\mathbf{p}+\mathbf{k}) - \beta)H, \mathbf{p}) - \beta H,$$

$$\tag{40}$$

$$\tilde{\varepsilon}_\downarrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\downarrow\downarrow}(\varepsilon_\downarrow(\mathbf{p}-\mathbf{k}) - \mu + \delta\omega - 2(\beta_1(\mathbf{p}-\mathbf{k}) - \beta)H, \mathbf{p}) + \beta H$$

Since we are interested only in smooth inhomogeneities with small $\mathbf{k}$ and, therefore, small $\delta\omega \propto k^2$, the small shifts of the arguments of the mass operators by $\delta\omega$ and $\mathbf{k}$ in the r.h.s. of Eq. (40) are not crucial. What really makes the *pseudo*-energies (40) different from the energies (37) is the term $2(\beta_1(\mathbf{p}) - \beta)H$ in the arguments of the mass operators. This term is small either at low polarizations (small H), or at low densities (β_1 is close to β). Only in these cases the *pseudo*-energies (40) are close to real energies (37); then the differences $\tilde{\varepsilon}_\downarrow(\mathbf{p}) - \varepsilon_\downarrow(\mathbf{p})$ and $\tilde{\varepsilon}_\uparrow(\mathbf{p}) - \varepsilon_\uparrow(\mathbf{p})$ are negligible, and *pseudo*-particles coincide with usual (quasi-)particles. What is more, in the main approximations for these limiting cases, the functions $W_\uparrow$ and $W_\downarrow$ become equal to each other, the matrix $\hat{F}$ degenerates, and the doubling of equations disappears reducing the system (34) to a single kinetic equation. We will discuss this in more detail in Parts II and III of this series.

As for a general case, Eqs. (35) can be written in a manner vaguely resembling the corresponding Silin equations:

$$
W_\downarrow(K;\mathbf{p}) = \left[\delta\omega + 2\beta H - \varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) + \tilde{\varepsilon}_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right) \right] Z_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right),
$$

$$
W_\uparrow(K;\mathbf{p}) = \left[\delta\omega + 2\beta H - \tilde{\varepsilon}_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) + \varepsilon_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right) \right] Z_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right),
$$

$$(41)$$

We want to emphasize here that the link between $\delta n_{\downarrow,\uparrow}(\mathbf{p})$ and densities of real particles with up and down spins or corresponding Landau quasi-particles is not trivial. This question, together with the closely related problem of macroscopic equations of motion, will be studied in detail elsewhere. One may consider the densities $\delta n_{\uparrow,\downarrow}(\mathbf{p})$ as partial contributions from the mixed spin components of slightly tilted spin-up and spin-down Landau quasi-particles to the overall off-diagonal (mixed) component of a single-particle density matrix, i.e. as partial contributions from up and down spins to the distribution of the transverse component $m^+(\mathbf{p})$ of magnetization $\mathbf{m}(\mathbf{p})$. Symbolically, it may be written as $\delta n_\uparrow(\mathbf{p}) = \delta n_{\uparrow\downarrow}^{(\uparrow)}(\mathbf{p})$, $\delta n_\downarrow(\mathbf{p}) = \delta n_{\uparrow\downarrow}^{(\downarrow)}(\mathbf{p})$. However, one should keep in mind that the states $\delta n_{\uparrow,\downarrow}(\mathbf{p})$ correspond to complex energies and are short-lived having a large attenuation at high polarizations/densities. Therefore, the partial transverse densities $\delta n_{\uparrow,\downarrow}(\mathbf{p})$ might also be considered not as some real quasi-particles' densities, but as some auxiliary quantities appearing on intermediate stages of microscopic calculations.

We also want to give here the relation between $\delta n_{\downarrow,\uparrow}(\mathbf{p})$ and the off-diagonal element $\delta N_{\downarrow\uparrow}(\mathbf{p})$ of the single-particle density matrix $\hat{N}(\mathbf{p})$ for real particles

$$
\delta N_\sigma(\mathbf{p}) = -i \int \delta G_\sigma(p_0,\mathbf{p}) \frac{dp_0}{2\pi}, \tag{42}
$$

which assumes the form

$$
\delta N_{\uparrow\downarrow}(\mathbf{p}) = \frac{1}{4} \int \frac{\left[G_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P-\frac{K}{2}\right) \right]}{\left(\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P+\frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P-\frac{K}{2}\right) \right)}
$$

$$
\times \left\{ \int \left[\mathfrak{S}(K;P,P'_\downarrow)\delta n_\downarrow(\mathbf{p}') \right. \right.
$$

$$
\left. \left. + \mathfrak{S}(K;P,P'_\uparrow)\delta n_\uparrow(\mathbf{p}') \right] \left[\theta_\downarrow(\mathbf{p}') - \theta_\uparrow(\mathbf{p}') \right] \frac{d^3p'}{(2\pi)^3} \right\} dp_0 \tag{43}
$$

If one would like to introduce quasi-paticles in a straightforward way, one would have to write an analogous equation but for an ideal gas of quasi-particles. In case of an ideal gas, the regular part ϕ_r of the difference of two Green's functions (20) is identically equal to zero, and the only remaining singular part ϕ_s contains δ-functions (30) allowing trivial integration over dp_0 in Eq. (43):

$$
\delta \tilde{N}_{\uparrow\downarrow}(\mathbf{p}) = \tfrac{1}{4}[\theta_\downarrow(\mathbf{p}) - \theta_\uparrow(\mathbf{p})] \int \left\{ \left[\frac{Z_\downarrow(\mathbf{p})}{Z_\downarrow(\mathbf{p}')} \frac{F_{\downarrow\downarrow}(\mathbf{p},\mathbf{p}')}{W_\downarrow(\mathbf{p})} \right. \right.
$$
$$
+ \frac{Z_\uparrow(\mathbf{p})}{Z_\downarrow(\mathbf{p}')} \frac{F_{\uparrow\downarrow}(\mathbf{p},\mathbf{p}')}{W_\uparrow(\mathbf{p})} \bigg] \delta n_\downarrow(\mathbf{p}')
$$
$$
+ \left[\frac{Z_\downarrow(\mathbf{p})}{Z_\uparrow(\mathbf{p}')} \frac{F_{\downarrow\uparrow}(\mathbf{p},\mathbf{p}')}{W_\downarrow(\mathbf{p})} + \frac{Z_\uparrow(\mathbf{p})}{Z_\uparrow(\mathbf{p}')} \frac{F_{\uparrow\uparrow}(\mathbf{p},\mathbf{p}')}{W_\uparrow(\mathbf{p})} \right] \delta n_\uparrow(\mathbf{p}') \bigg]
$$
$$
\times [\theta_\downarrow(\mathbf{p}') - \theta_\uparrow(\mathbf{p}')] \bigg\} \frac{d^3 p'}{(2\pi)^3} \tag{44}
$$

Now it is time to determine the spin-wave spectrum from Eqs. (34). We can do it by expanding Eqs. (34) in $\mathbf{k}$. This procedure is equivalent to a gradient expansion for the equations of motion. For exactly the same reasons as in Sec. 2, it is sufficient to leave only the first two terms in the expansion of the transverse densities in $\mathbf{k}$:

$$
\delta n_{\downarrow,\uparrow}(\mathbf{p}) = \delta n_{\downarrow,\uparrow}^{(0)}(\mathbf{p}) + \mathbf{p}\mathbf{k}\, \delta n_{\downarrow,\uparrow}^{(1)}(\mathbf{p}) \tag{45}
$$

and to disregard the term $n_{\downarrow,\uparrow}^{(2)}(\mathbf{p})$. According to Eqs. (10), (22) and (33),

$$
\delta n_\downarrow^{(0)}(\mathbf{p}) = A, \qquad \delta n_\uparrow^{(0)}(\mathbf{p}) = A \tag{46}
$$

where A is an irrelevant constant (generically the same as in Eq. (10)) characterizing the amplitude of perturbations in homogeneous equations; below we assume $A = 1$.

Equation for $\delta n_{\downarrow,\uparrow}^{(1)}(\mathbf{p})$ is obtained by substituting Eqs. (45), (46) into Eq. (34) and extracting the linear in $\mathbf{k}$ term from this equation:

$$
W_\downarrow(K_0; \mathbf{p}) \delta n_\downarrow^{(1)}(\mathbf{p}) - \frac{1}{2} \int \frac{\mathbf{p}'\mathbf{k}}{\mathbf{p}\mathbf{k}} [F_{\downarrow\downarrow}(K_0; \mathbf{p},\mathbf{p}') \delta n_\downarrow^{(1)}(\mathbf{p}')
$$
$$
+ F_{\downarrow\uparrow}(K_0; \mathbf{p},\mathbf{p}') \delta n_\uparrow^{(1)}(\mathbf{p}')][\theta(\varepsilon_\downarrow(\mathbf{p}') - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p}') - \mu)] \frac{d^3 p'}{(2\pi)^3}
$$
$$
= - W_{\downarrow k}(K_0; p) + \frac{1}{2} \int \{[F_{\downarrow\downarrow k}(K_0; \mathbf{p},\mathbf{p}') + F_{\downarrow\uparrow k}(K_0; \mathbf{p},\mathbf{p}')]
$$
$$
\times [\theta(\varepsilon_\downarrow(\mathbf{p}') - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p}') - \mu)] + \tfrac{1}{2}[F_{\downarrow\downarrow}(K_0; \mathbf{p},\mathbf{p}') + F_{\downarrow\uparrow}(K_0; \mathbf{p},\mathbf{p}')]
$$
$$
\times [\delta(\varepsilon_\downarrow(\mathbf{p}') - \mu)\mathbf{v}_\downarrow(\mathbf{p}') + \delta(\varepsilon_\uparrow(\mathbf{p}') - \mu)\mathbf{v}_\uparrow(\mathbf{p}')]\} \frac{d^3 p'}{(2\pi)^3}, \tag{47}
$$

$$W_\uparrow(K_0; \mathbf{p})\delta n_\uparrow^{(1)}(\mathbf{p}) - \frac{1}{2}\int \frac{\mathbf{p'k}}{\mathbf{pk}} [F_{\uparrow\downarrow}(K_0; \mathbf{p}, \mathbf{p'})\delta n_\downarrow^{(1)}(\mathbf{p'})$$

$$+ F_{\uparrow\uparrow}(K_0; \mathbf{p}, \mathbf{p'})\delta n_\uparrow^{(1)}(\mathbf{p'})][\theta(\varepsilon_\downarrow(\mathbf{p'}) - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p'}) - \mu)]\frac{d^3 p'}{(2\pi)^3}$$

$$= -W_{\uparrow k}(K_0; \mathbf{p}) + \frac{1}{2}\int \{[F_{\uparrow\downarrow k}(K_0; \mathbf{p}, \mathbf{p'}) + F_{\uparrow\uparrow k}(K_0; \mathbf{p}, \mathbf{p'})]$$

$$\times[\theta(\varepsilon_\downarrow(\mathbf{p'}) - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p'}) - \mu)] + \tfrac{1}{2}[F_{\uparrow\downarrow}(K_0; \mathbf{p}, \mathbf{p'}) + F_{\uparrow\uparrow}(K_0; \mathbf{p}, \mathbf{p'})$$

$$\times[\delta(\varepsilon_\downarrow(\mathbf{p'}) - \mu)\nu_\downarrow(\mathbf{p'}) + \delta(\varepsilon_\uparrow(\mathbf{p'}) - \mu)\nu_\uparrow(\mathbf{p'})]\}\frac{d^3 p'}{(2\pi)^3},$$

where we used the following notations:

$$W_{\downarrow,\uparrow k}(K; \mathbf{p})\mathbf{pk} = \frac{\partial W_{\downarrow,\uparrow}(K; \mathbf{p})}{\partial \mathbf{k}}\mathbf{k},$$

$$F_{\downarrow\downarrow k}(K; \mathbf{p}, \mathbf{p'})\mathbf{pk} = \frac{\partial F_{\downarrow\downarrow}(K; \mathbf{p}, \mathbf{p'})}{\partial \mathbf{k}}\mathbf{k}, \qquad K_0 = (2\beta H, 0), \qquad (48)$$

and the same for other components of the matrix $\hat{F}$. The presence of the θ-functions in the integrands in Eqs. (47) makes these equations very difficult to solve analytically in a general case; the terms with δ-functions are much simpler since they effectively eliminate the momentum integration and reduce the corresponding part of the equation to an angular problem on the Fermi surfaces. The formal exact solution of Eqs. (47) at arbitrary polarizations will be given in Part III.

Expanding Eqs. (34) to the next order in $\mathbf{k}$ and integrating it over $\mathbf{p}$, we obtain the following expression for α:

$$\alpha = \frac{Z_{\text{eff}}}{N_\uparrow - N_\downarrow}\int \left\{ -W_{\downarrow\mathbf{p}}(\mathbf{p})\delta n_\downarrow^{(1)}(\mathbf{p})\frac{(\mathbf{pk})^2}{4k^2} - W_{\uparrow\mathbf{p}}(\mathbf{p})\delta n_\uparrow^{(1)}(\mathbf{p})\frac{(\mathbf{pk})^2}{4k^2} \right.$$

$$+ \frac{1}{4}\int \left\{ \left[F_{\downarrow\downarrow k_i k_j}(K_0; \mathbf{p}, \mathbf{p'})\frac{k_i k_j}{2k^2} + F_{\downarrow\uparrow k_i k_j}(K_0; \mathbf{p}, \mathbf{p'})\frac{k_i k_j}{2k^2} \right. \right.$$

$$+ F_{\uparrow\downarrow k_i k_j}(K_0; \mathbf{p}, \mathbf{p'})\frac{k_i k_j}{2k^2} + F_{\uparrow\uparrow k_i k_j}(K_0; \mathbf{p}, \mathbf{p'})\frac{k_i k_j}{2k^2}$$

$$+ F_{\downarrow\downarrow k}(K_0; \mathbf{p}, \mathbf{p'})\delta n_\downarrow^{(1)}(\mathbf{p'})\frac{(\mathbf{pk})(\mathbf{p'k})}{k^2}$$

$$+ F_{\downarrow\uparrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\uparrow}^{(1)}(\mathbf{p}') \frac{(\mathbf{pk})(\mathbf{p'k})}{k^2}$$

$$+ F_{\downarrow\uparrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\downarrow}^{(1)}(\mathbf{p}') \frac{(\mathbf{pk})(\mathbf{p'k})}{k^2}$$

$$+ F_{\uparrow\uparrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\uparrow}^{(1)}(\mathbf{p}') \frac{(\mathbf{pk})\mathbf{p'k})}{k^2} \Bigg]$$

$$\times [\theta(\varepsilon_{\downarrow}(\mathbf{p}') - \mu) - \theta(\varepsilon_{\uparrow}(\mathbf{p}') - \mu)] + \Bigg[F_{\downarrow\downarrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\frac{\mathbf{pk}}{k}$$

$$+ F_{\downarrow\uparrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\frac{\mathbf{pk}}{k} + F_{\uparrow\downarrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\frac{\mathbf{pk}}{k} + F_{\uparrow\uparrow\mathbf{k}}(K_0; \mathbf{p}, \mathbf{p}')\frac{\mathbf{pk}}{k}$$

$$+ F_{\downarrow\downarrow}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\downarrow}^{(1)}(\mathbf{p}')\frac{\mathbf{p'k}}{k} + F_{\downarrow\uparrow}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\uparrow}^{(1)}(\mathbf{p}')\frac{\mathbf{p'k}}{k}$$

$$+ F_{\uparrow\downarrow}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\downarrow}^{(1)}(\mathbf{p}')\frac{\mathbf{p'k}}{k} + F_{\uparrow\uparrow}(K_0; \mathbf{p}, \mathbf{p}')\delta n_{\uparrow}^{(1)}(\mathbf{p}')\frac{\mathbf{p'k}}{k} \Bigg]$$

$$\times \Bigg[\delta(\varepsilon_{\downarrow}(\mathbf{p}') - \mu)\frac{\mathbf{\nu}_{\downarrow}(\mathbf{p}')\mathbf{k}}{k} + \delta(\varepsilon_{\uparrow}(\mathbf{p}') - \mu)\frac{\mathbf{\nu}_{\uparrow}(\mathbf{p}')\mathbf{k}}{k} \Bigg] \frac{d^3p'}{(2\pi)^3} \Bigg\}$$

$$\times [\theta(\varepsilon_{\downarrow}(\mathbf{p}) - \mu) - \theta(\varepsilon_{\uparrow}(\mathbf{p}) - \mu)] \frac{d^3p}{(2\pi)^3} \qquad (49)$$

where

$$Z_{eff}^{-1} = \frac{1}{N_{\uparrow} - N_{\downarrow}} \int \Bigg\{ 1 - \frac{1}{2}\Bigg[\Sigma_{\downarrow\downarrow p_0}\Big(P_{\uparrow} + \frac{K_0}{2}\Big) + \Sigma_{\uparrow\uparrow p_0}\Big(P_{\downarrow} - \frac{K_0}{2}\Big) \Bigg]$$

$$- \frac{1}{2}\int [F_{\downarrow\downarrow k_0}(K_0; \mathbf{p}, \mathbf{p}')$$

$$+ F_{\downarrow\uparrow k_0}(K_0; \mathbf{p}, \mathbf{p}') + F_{\uparrow\downarrow k_0}(K_0; \mathbf{p}, \mathbf{p}') + F_{\uparrow\uparrow k_0}(K_0; \mathbf{p}, \mathbf{p}')]$$

$$\times [\theta(\varepsilon_{\downarrow}(\mathbf{p}') - \mu) - \theta(\varepsilon_{\uparrow}(\mathbf{p}') - \mu)] \frac{d^3p'}{(2\pi)^3} \Bigg\}$$

$$\times [\theta(\varepsilon_{\downarrow}(\mathbf{p}) - \mu) - \theta(\varepsilon_{\uparrow}(\mathbf{p}) - \mu)] \frac{d^3p}{(2\pi)^3}, \qquad (50)$$

$$\mathbf{\nu}_{\downarrow,\uparrow}(\mathbf{p}) = \frac{\partial \varepsilon_{\downarrow,\uparrow}(\mathbf{p})}{\partial \mathbf{p}}, \qquad m_{\downarrow,\uparrow}(\mathbf{p})\delta_{ij} = \frac{\partial^2 \varepsilon_{\downarrow,\uparrow}(\mathbf{p})}{\partial p_i \partial p_j},$$

$$F_{\downarrow\downarrow k_i k_j}(K_0; \mathbf{p}) = \frac{\partial^2 F_{\downarrow\downarrow}(K_0; \mathbf{p})}{\partial k_i \partial k_j},$$

and the same for other "arrow" components of F and W.

Let us also transform Eqs. (8), (9) to the above set of variables. These equations are practically identical, and correspond to Eqs. (34) with $K = K_0$ (48):

$$W_\downarrow(\mathbf{p}) = \frac{1}{2} \int [F_{\downarrow\downarrow}(\mathbf{p}, \mathbf{p}') + F_{\downarrow\uparrow}(\mathbf{p}, \mathbf{p}')]$$

$$\times [\theta(\varepsilon_\downarrow(\mathbf{p}') - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p}') - \mu)] \frac{d^3 p'}{(2\pi)^3},$$

$$W_\uparrow(\mathbf{p}) = \frac{1}{2} \int [F_{\uparrow\downarrow}(\mathbf{p}, \mathbf{p}') + F_{\uparrow\uparrow}(\mathbf{p}, \mathbf{p}')]$$

$$\times [\theta(\varepsilon_\downarrow(\mathbf{p}') - \mu) - \theta(\varepsilon_\uparrow(\mathbf{p}') - \mu)] \frac{d^3 p'}{2(2\pi)^3},$$

(51)

Here we introduced the notations

$$W_i(\mathbf{p}) = W_i(K_0; \mathbf{p}), \qquad F_{ik}(\mathbf{p}, \mathbf{p}') = F_{ik}(K_0; \mathbf{p}, \mathbf{p}'), \qquad \{i, k\} = \{\uparrow, \downarrow\} \quad (52)$$

and used Eq. (46).

4. PHENOMENOLOGICAL EQUATIONS OF TRANSVERSE SPIN DYNAMICS

We want to establish links with a standard phenomenological Landau theory, and to develop a macroscopic version of the above equations. The Landau description of Fermi liquids is based (for details see Refs. 18, 22) on a concept of a nearly ideal gas of quasi-particles with a mean field Landau interaction. This interaction is proportional to the deviation of the quasi-particles' distribution $\delta n_\sigma(\mathbf{p})$ from equilibrium, and is determined by the Landau interaction function $F_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')$. The Landau interaction can be best described by adding an additional term to the phenomenological quasi-particles' Hamiltonian:

$$\delta\varepsilon_\sigma(\mathbf{p}) = \text{Tr}_{\sigma'} \int F_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}') \delta n_{\sigma'} \frac{d^3 p}{(2\pi)^3} \qquad (53)$$

where $\delta\varepsilon_\sigma(\mathbf{p})$ is the change in the energy of quasi-particles with spin σ as a result of change in the distribution function of quasi-particles $\delta n_{\sigma'}(\mathbf{p}')$, and the spin structure of the Landau interaction function has the following simple form in the absence of polarization

$$F_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}') = \tfrac{1}{2} F^s(\mathbf{p}, \mathbf{p}') + \tfrac{1}{2} \sigma\sigma' F^a(\mathbf{p}, \mathbf{p}') \qquad (54)$$

Spin polarization of a Fermi liquid results in certain obvious and some more subtle changes. One of these changes is a definite non-locality of the

theory (see comments in Refs. 2, 6), especially if one is interested in transverse spin dynamics up to the terms of the order k^2. The corresponding generalization of the theory is straightforward: Eq. (53) should be substituted by a non-local expansion of the single-particle Hamiltonian in changes of density distributions $\delta n_\sigma(\mathbf{p}, \mathbf{r})$ and their spatial derivatives:

$$\delta\varepsilon_\sigma(\mathbf{p}, \mathbf{r}) = \mathrm{Tr}_{\sigma'} \int F_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')\delta n_{\sigma'}(\mathbf{p}', \mathbf{r}) \frac{d^3 p'}{(2\pi)^3}$$

$$+ \mathrm{Tr}_{\sigma'} \int F^{(i)}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')\nabla_i \delta n_{\sigma'}(\mathbf{p}', \mathbf{r}) \frac{d^3 p'}{(2\pi^3)}$$

$$+ \mathrm{Tr}_{\sigma'} \int F^{(ik)}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')\nabla_i \nabla_k \delta n(\mathbf{p}', \mathbf{r}) \frac{d^3 p'}{(2\pi)^3} + \ldots \quad (55)$$

where the "additional" Landau functions $F^{(i)}$ and $F^{(ik)}$ may be either considered as some new phenomenological interaction functions or expressed through the vertex part and its harmonics. Generally speaking, the gradient expansion (55) should contain all derivatives of the distribution. However, we are not interested, at least in this particular context, in transverse phenomena beyond the k^2 accuracy—the higher order terms in $\mathbf{k}$ should be always neglected if one wants to maintain some link between a longwave microscopic description and some semi-classical macroscopic ("hydrodynamic") approach to magnetic phenomena. Therefore, it is reasonable to truncate the expansion (55) after the second-order gradients.

Since the time and spatial dependencies of the distribution $\delta n_\sigma(\mathbf{p}, \mathbf{r}, t)$ are the exponential ones (see Eq. (6)), the equation (55) can be rewritten as

$$\delta\varepsilon_\sigma(\mathbf{p}) = \mathrm{Tr}_{\sigma'} \int \tilde{F}_{\sigma,\sigma'}\left(\mathbf{p}+\frac{\mathbf{k}}{2}, \mathbf{p}'-\frac{\mathbf{k}}{2}\right)\delta n_{\sigma'}(\mathbf{p}') \frac{d^3 p'}{(2\pi)^3} \quad (56)$$

where the derivatives of the function $\tilde{F}$ are equal to the above functions $F^{(i)}$ and $F^{(ik)}$:

$$F^{(i)}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}') = -\frac{i}{2}\left(\frac{\partial \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p_i} - \frac{\partial \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p'_i}\right),$$

$$F^{(ik)}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}') = \frac{1}{8}\left(\frac{\partial^2 \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p_i \, \partial p_k} + \frac{\partial^2 \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p'_k \, \partial p'_i}\right.$$

$$\left. - \frac{\partial^2 \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p'_i \, \partial p_k} - \frac{\partial^2 \tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}')}{\partial p_i \, \partial p'_k}\right) \quad (57)$$

In case of polarized systems, the spin structure of the non-local interaction operator $\tilde{F}_{\sigma,\sigma'}$ is different from a simple form (54)[1]:

$$\tilde{F}_{\sigma,\sigma'}(\mathbf{p}, \mathbf{p}') = \tilde{F}_1(\mathbf{p}, \mathbf{p}') + \tilde{F}_2(\mathbf{p}, \mathbf{p}')\sigma\sigma' + \{\tilde{F}_3(\mathbf{p}, \mathbf{p}')\sigma e + \tilde{F}_3(\mathbf{p}', \mathbf{p})\sigma' e\}$$

$$+ \tilde{F}_4(\mathbf{p}, \mathbf{p}')(\sigma e)(\sigma' e) \tag{58}$$

where e is the unit vector in the direction of the equilibrium spin polarization (usually, the z-direction in spin space).

Eqs. (56) (or (55)) and (58) should be substituted into the semi-classical Landau kinetic equations for the quasi-particles,

$$\frac{\partial \hat{n}}{\partial t} - i[\hat{n}, \hat{\varepsilon}] = 0 \tag{59}$$

where the linearized quantum commutator,

$$[\hat{A} + \delta\hat{A}, \hat{B} + \delta\hat{B}]_{\sigma\sigma'} = \delta A_{\sigma\sigma''}(\mathbf{p}) B^{(0)}_{\sigma''\sigma'}\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) - B^{(0)}_{\sigma\sigma''}\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)\delta A_{\sigma''\sigma'}(\mathbf{p})$$

$$- \delta B_{\sigma\sigma''}(\mathbf{p}) A^{(0)}_{\sigma''\sigma'}\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) + A^{(0)}_{\sigma\sigma''}\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)\delta B_{\sigma''\sigma'}(\mathbf{p}) \tag{60}$$

includes both the spin commutator and the semi-classical Poisson bracket. The resulting equation is usually supposed to give a correct macroscopic picture of longitudinal (pure spin components of $\hat{n}$) and transverse (mixed spin components of $\hat{n}$) macroscopic phenomena in spin-polarized Fermi liquids.

Unfortunately, this simple picture does not correspond to the above microscopic equations, and fails to give a proper macroscopic description for transverse phenomena. The transverse dynamics cannot be described by a single kinetic equation of the type (56)–(60) in mixed spin components of some single distribution function $\hat{n}$, but, instead should be described with the help of two coupled kinetic equations of the type (34) in two-component distribution function. Therefore, the proper phenomenological theory should be constructed along the same lines as it is sometimes done[1] for longitudinal phenomena using an analogy between a spin-polarized system and a binary mixture of spin-up and spin-down components.

For longitudinal processes, i.e. for diagonal components of the single-particle density matrix, such an analogy is quite natural and very simple. We do have two sorts of (quasi-) particles, $N_{\uparrow\uparrow}(\mathbf{p})$ and $N_{\downarrow\downarrow}(\mathbf{p})$, with different

energy spectra, $\varepsilon_{\uparrow\uparrow}(\mathbf{p})$ and $\varepsilon_{\downarrow\downarrow}(\mathbf{p})$, and with different equations of motion coupled only through the mean field Landau interaction (and, at finite temperatures, through the collision operator). These (quasi-)particles remain near the corresponding Fermi surfaces throughout the interaction processes, while their total numbers remain constant.

For some reasons, this analogy has not been used before for a description of transverse effects, i.e. for off-diagonal states of (quasi-) particles $N_{\uparrow\downarrow}(\mathbf{p})$. Here one usually assumes that there is only one type of off-diagonal states. This would be true if one would deal with bare non-interacting particles (say, in a dilute gas). Then it would be of no interest what sort of particles (spin-up or spin-down) acquires off-diagonal spin components by flipping onto xy-plane. This is not necessarily so for dressed particles: these particles are dressed in a different way depending on the spin projection, and the energy/response may be different depending on which sort of particles got flipped even if the resulting density perturbation $\delta N_{\uparrow\downarrow}(\mathbf{p})$ is the same.

One can put it in a different way. In case of bare non-interacting particles, one can obtain a spin-down particle with some momentum $\mathbf{p}$ and energy spectrum $\varepsilon_{\uparrow}(\mathbf{p})$ by simply turning over a spin-up particle with the same momentum. However, when one changes the spin projection of a dressed (quasi-) particle with momentum $\mathbf{p}$ in a strongly interacting polarized system, one ends up with a (quasi-) particle with some energy $\tilde{\varepsilon}_{\downarrow}(\mathbf{p})$ different from expected $\varepsilon_{\downarrow}(\mathbf{p})$.

Transverse spin dynamics corresponds to dynamics of (small) xy-components of magnetization when the magnetization is slightly tilted from its equilibrium z-direction. When the magnetization is in equilibrium (untilted), we have two types of particles—spin-ups and spin-downs in pure spin states. If one tilts the magnetization, one will also have particles in off-diagonal mixed spin states. The usual picture of off-diagonal particles presents them as particles oscillating between upper and lower energy levels corresponding to pure states. In case of dressed Landau quasi-particles, the usual concept of a mixed state is a quasi-particle oscillating between Fermi surfaces for up and down spins. However, we have seen in the previous section, that we have two types of mixed quasi-particle states: one oscillating from the upper Fermi surface down to the surface shifted from it by the bare Larmor frequency $2\beta H$ and back, and another oscillating from the lower Fermi surface up to the surface shifted from it also by the bare Larmor frequency and back. Since the distance between the Fermi surfaces for *dressed* spin-up and spin-down particles (39) $2\beta_1 H$ is very different from the Larmor frequency $2\beta H$, the frequencies of these two types of oscillations are different. What is more, it is obvious that the molecular fields acting on these two types of oscillating off-diagonal states should also be different.

Therefore, when we slightly tilt the direction of magnetization, we end up with two types of oscillations: for initially spin-up particles—between the state with the spectra $\varepsilon_\uparrow(\mathbf{p})$ and the *pseudo*-state $\tilde{\varepsilon}_\downarrow(\mathbf{p})$, and for spin-down quasi-particle—between the state $\varepsilon_\downarrow(\mathbf{p})$ and the *pseudo*-state $\tilde{\varepsilon}_\uparrow(\mathbf{p})$. These four types of states are defined in a different way by mass operators (37), (38):

$$\varepsilon_\uparrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\uparrow\uparrow}(\varepsilon_\uparrow(\mathbf{p}) - \mu, \mathbf{p}) - \beta H,$$

$$\tilde{\varepsilon}_\uparrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\uparrow\uparrow}(\varepsilon_\downarrow(\mathbf{p}+\mathbf{k}) - \mu - \delta\omega - 2\beta H, \mathbf{p}) - \beta H,$$

$$\varepsilon_\downarrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\downarrow\downarrow}(\varepsilon_\downarrow(\mathbf{p}) - \mu, \mathbf{p}) + \beta H,$$

$$\tilde{\varepsilon}_\downarrow(\mathbf{p}) = \frac{p^2}{2m} + \Sigma_{\downarrow\downarrow}(\varepsilon_\uparrow(\mathbf{p}-\mathbf{k}) - \mu + \delta\omega + 2\beta H, \mathbf{p}) + \beta H,$$

(61)

In case of strongly interacting particles and noticeable polarizations, the spectra $\varepsilon(\mathbf{p})$ and $\tilde{\varepsilon}(\mathbf{p})$ are definitely different. The mutual dephasing of these two types of oscillations is responsible in inhomogeneous situations for a noticeable inhomogeneous broadening resulting effectively in a zero-temperature attenuation.

The above reasoning may sound rather vague if presented on a purely phenomenological ground. Nevertheless, we have a sound microscopic justification for separating off-diagonal perturbations by their origin. The exact microscopic transverse equations explicitly involve the transverse Green's function at two distinctly separate frequencies shifted by $\Omega_0 = 2\beta H$ thus resulting in a strong temporal non-locality. There is no way one can rewrite these equations using only one density matrix which has a single time variable and corresponds to a single-frequency Green's function.

Having established the existence of two types of off-diagonal states, we must write the "phenomenological" equations of motion for this objects in a spirit of Eqs. (56)–(60), but in a form reducible to exact microscopic equations (34), (41). One can do it in a several equivalent ways. The simplest and the most straightforward one would be just to introduce outright two phenomenological transverse densities oscillating between two pairs of states (61), and coupled via a 4-component non-local Landau operator.

We feel that a better, though a slightly more cumbersome, way of introducing equations for phenomenological quasi-particles, is to introduce an auxilliary *pseudo*-spin operator $\hat{\rho}$ which has the structure of standard Pauli matrices and whose eigenvalues separate real quasi-particles' states and *pseudo*-states (the corresponding unitary operator will be denoted as

$\hat{I}$, while the unitary operator in the usual spin space will be $\hat{\delta}$). The spin and *pseudo*-spin operators $\hat{\sigma}$ and $\hat{\rho}$ operate in different spaces and, therefore, commute with each other.

In equilibrium one has only spin-up and spin-down particles with the distributions $N_{\uparrow\uparrow}(\mathbf{p}) = 1 - \theta(\varepsilon_\uparrow(\mathbf{p}) - \mu)$ and $N_{\downarrow\downarrow}(\mathbf{p}) = 1 - \theta(\varepsilon_\downarrow(\mathbf{p}) - \mu)$, and no *pseudo*-particles. Therefore, the equilibrium distribution can be written as

$$\hat{n}^{(0)}_{\sigma,\rho}(\mathbf{p}) = \tfrac{1}{4}\hat{I}[(2 - \theta(\varepsilon_\uparrow - \mu) - \theta(\varepsilon_\downarrow - \mu))\hat{\delta} + (\theta(\varepsilon_\downarrow - \mu) - \theta(\varepsilon_\uparrow - \mu))\sigma^z] \quad (62)$$

while the equilibrium single-particle Hamiltonian is also diagonal,

$$\hat{H}^{(0)}_{\sigma,\rho}(\mathbf{p}) = \tfrac{1}{4}(I + \hat{\rho}^z)[(\varepsilon_\uparrow + \tilde{\varepsilon}_\downarrow)\hat{\delta} + (\varepsilon_\uparrow - \tilde{\varepsilon}_\downarrow)\hat{\sigma}^z]$$

$$+ \tfrac{1}{4}(I - \hat{\rho}^z)[(\tilde{\varepsilon}_\uparrow + \varepsilon_\downarrow)\hat{\delta} + (\tilde{\varepsilon}_\uparrow - \varepsilon_\downarrow)\hat{\sigma}^z] \quad (63)$$

where the arrow indices correspond to up and down spin states. Of course, since both $\hat{n}^{(0)}$ and $\hat{H}^{(0)}$ do not depend on coordinates, the full quantum commutator (59), $[\hat{n}^{(0)}, \hat{H}^{(0)}] = 0$. When the particle distribution deviates from equilibrium, the change of the single-particle operator will be determined by some molecular field interaction, $\hat{F}$, which should be defined as a two-particle operator in both spin and *pseudo*-spin spaces:

$$\delta\hat{H}_{\sigma,\rho}(\mathbf{p}) = \mathrm{Tr}_{\sigma'}\,\mathrm{Tr}_{\rho'} \int \frac{d^3p'}{(2\pi)^3} \hat{F}_{\sigma\sigma',\rho\rho'}(\mathbf{k}; \mathbf{p}, \mathbf{p}')\delta\hat{n}_{\sigma',\rho'}(\mathbf{p}') \quad (64)$$

where the presence of the vector $\mathbf{k}$ in the argument of the interaction operator takes into account the non-locality of the interaction. The spin structure of the operator $\hat{F}$ is the same as in Eq. (58).

There is no equilibrium polarization in the *pseudo*-spin space which results in a diagonal *pseudo*-spin structure of the operator $\hat{F}$ (i.e., the *pseudo*-spin structure of the functions F_i in Eq. (58)):

$$\hat{F}_{i,\rho\rho'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') = F_i^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')\hat{I}\hat{I}' + F_i^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')\hat{\rho}\hat{\rho}'$$

$$+ F_i^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')\hat{I}\hat{\rho}^{z'} + F_i^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})\hat{\rho}^z\hat{I}' \quad (65)$$

The general kinetic equations (59), (60) corresponding to the quasi-particles (63)–(65) are very cumbersome. Therefore, in this particular context it is reasonable to restrict ourselves to the perturbation of the spin-wave type which is off-diagonal in the spin space being proportional to $\hat{\sigma}^+ = \hat{\sigma}^x + i\hat{\sigma}^y$, and is diagonal in the *pseudo*-spin space:

$$\delta\hat{n}_{\sigma,\rho}(\mathbf{p}) = \hat{\sigma}^+[(\delta n_\uparrow(\mathbf{p}) + \delta n_\downarrow(\mathbf{p}))\hat{I} + (\delta n_\uparrow(\mathbf{p}) - \delta n_\downarrow(\mathbf{p}))\hat{\rho}^z] \quad (66)$$

where now the arrow indices correspond to the up and down *pseudo*-spin

states. The corresponding change in energy is equal to

$$\delta \hat{H}_{\sigma,\rho}(\mathbf{k}; \mathbf{p}) = \hat{\sigma}^{+} \int \frac{d^3p'}{(2\pi)^3} [(\hat{I} F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + \hat{\rho}^z F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')$$

$$+ \hat{I} F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + \hat{\rho}^z F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\uparrow$$

$$+ (\hat{I} F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - \hat{\rho}^z F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - \hat{I} F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')$$

$$+ \hat{\rho}^z F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\downarrow] \tag{67}$$

The full commutator (59), (60) contains a nearly standard dynamic part,

$$[\delta \hat{n}, \hat{H}^{(0)}] = -\frac{\hat{\sigma}^{+}}{2} \left\{ \hat{I} \left[\delta n_\uparrow(\mathbf{p}) \tilde{\varepsilon}_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) - \varepsilon_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) \delta n_\uparrow(\mathbf{p}) \right. \right.$$

$$\left. + \delta n_\downarrow(\mathbf{p}) \varepsilon_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) - \tilde{\varepsilon}_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) \delta n_\downarrow(\mathbf{p}) \right]$$

$$+ \hat{\rho}^z \left[\delta n_\uparrow(\mathbf{p}) \tilde{\varepsilon}_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) - \varepsilon_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) \delta n_\uparrow(\mathbf{p}) \right.$$

$$\left. \left. - \delta n_\downarrow(\mathbf{p}) \varepsilon_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) + \tilde{\varepsilon}_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) \delta n_\downarrow(\mathbf{p}) \right] \right\} \tag{68}$$

and a Fermi liquid part,

$$[\hat{n}^{(0)}, \delta \hat{H}] = -\frac{\sigma^{+}}{2} \int \frac{d^3p'}{(2\pi)^3} \left\{ \left[\theta \left(\varepsilon_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) - \mu \right) - \theta \left(\varepsilon_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) - \mu \right) \right] \right.$$

$$\times [(\hat{I} F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + \hat{\rho}^z F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')$$

$$+ \hat{I} F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + \hat{\rho}^z F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\uparrow(\mathbf{p}')$$

$$+ (\hat{I} F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - \hat{\rho}^z F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}')$$

$$\left. - \hat{I} F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + \hat{\rho}^z F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\downarrow(\mathbf{p}')] \tag{69} \right.$$

Finally, the matrix kinetic equation (59), (60), (68), (69) can be written in a form of two coupled equations:

$$\left(\omega + \varepsilon_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) - \tilde{\varepsilon}_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) \right) \delta n_\uparrow(\mathbf{p})$$

$$= \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} \left[\theta(\varepsilon_\downarrow \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) - \mu) - \theta \left(\varepsilon_\uparrow \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) - \mu \right) \right]$$

$$\times [(F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\uparrow(\mathbf{p}')$$

$$+ (F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p})) \delta n_\downarrow(\mathbf{p}')]$$

$$\left(\omega - \varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) + \tilde{\varepsilon}_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)\right)\delta n_\downarrow(\mathbf{p})$$

$$= \frac{1}{2}\int\frac{d^3p'}{(2\pi)^3}\left[\theta(\varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)-\mu)-\theta\left(\varepsilon_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)-\mu\right)\right]$$

$$\times[(F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}))\delta n_\uparrow(\mathbf{p}')$$

$$+ (F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}))\delta n_\downarrow(\mathbf{p}')]$$

$$(70)$$

Comparison of Eqs. (70) with exact microscopic equations (34), (45) shows that the above phenomenological picture coincides with the microscopic description developed in the previous section. The only visible difference concerns the slightly different ways in which the factors $\theta_\downarrow - \theta_\uparrow$ enter the equations. However, this deviation is not at all significant, and can be easily taken care of by redefining the variables $\delta n_{\downarrow,\uparrow}$ by including these factors into them, $\delta n_{\downarrow,\uparrow} = (\theta_\downarrow - \theta_\uparrow)\delta\tilde{n}_{\uparrow,\downarrow}$. As a result of such a transformation, Eqs. (70) assume the form (34) with

$$F_{\uparrow\uparrow}(K; \mathbf{p}, \mathbf{p}') = F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}),$$

$$F_{\uparrow\downarrow}(K; \mathbf{p}, \mathbf{p}') = F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}),$$
$$(71)$$

$$F_{\downarrow\uparrow}(K; \mathbf{p}, \mathbf{p}') = F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}),$$

$$F_{\downarrow\downarrow}(K; \mathbf{p}, \mathbf{p}') = F_2^{(1)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') + F_2^{(2)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}, \mathbf{p}') - F_2^{(3)}(\mathbf{k}; \mathbf{p}', \mathbf{p}).$$

One can also add here the equations analogous to Eqs. (51),

$$(\omega + \varepsilon_\uparrow(\mathbf{p}) - \tilde{\varepsilon}_\downarrow(\mathbf{p})) = \frac{1}{2}\int\frac{d^3p'}{(2\pi)^3}[\theta(\varepsilon_\downarrow(\mathbf{p})-\mu) - \theta(\varepsilon_\uparrow(\mathbf{p})-\mu)]$$

$$\times[(F_{\uparrow\uparrow}(\mathbf{p}, \mathbf{p}') + F_{\uparrow\downarrow}(\mathbf{p}, \mathbf{p}')]$$

$$(\omega - \varepsilon_\downarrow(\mathbf{p}) + \tilde{\varepsilon}_\uparrow(\mathbf{p})) = \frac{1}{2}\int\frac{d^3p'}{(2\pi)^3}[\theta(\varepsilon_\downarrow(\mathbf{p})-\mu) - \theta(\varepsilon_\uparrow(\mathbf{p})-\mu)]$$

$$\times[(F_{\downarrow\uparrow}(\mathbf{p}, \mathbf{p}') + F_{\downarrow\downarrow}(\mathbf{p}, \mathbf{p}')]   (72)$$

which correspond to a homogeneous $\mathbf{k} = 0$ rotation of magnetization, and which ensure that the eigenvalue of Eqs. (70) at $\mathbf{k} = 0$ is the bare Larmor frequency $2\beta H$.

In essence, this completes the phenomenological description, and confirms the validity of the picture of transverse phenomena based on a doubling of relevant states (spin-up and spin-down states and *pseudo*-states).

The remaining difference between the microscopic and phenomenological approaches concerns some residual temporal non-locality of microscopic equations which manifests itself in a presence of the frequency $\delta\omega$ in the argument $K = (\delta\omega + 2\beta H, k)$ of the functions in the l.h.s. of Eqs. (71) and (and, actually, in the energy spectra of the *pseudo*-particles (61)). This can be reconciled only by a direct introduction of $\delta\omega$ into the arguments of phenomenological functions.

5. SUMMARY

The main result of this paper is a consistent theory of transverse dynamics in polarized Fermi liquid based on exact microscopic equations (Dyson equations (1), (2) at $T = 0$). The only assumption is an existence of a pole of Green's function for a *pure* spin state on the corresponding Fermi surface. Another very important feature of the theory is its intrinsic spatial and temporal non-locality.

An attempt to rewrite the exact equations in Green's functions (7) as equations in distribution functions of some quasi-particles (34), in a spirit of the Landau theory, results in a necessity of introducing two types of *transverse* quasi-particles (33)—generically, the transverse quasi-particles originating from spin-up and spin-down quasi-particles with slightly tilted spins. The dynamics of these two types of transverse quasi-particles is associated, bascially, with oscillations between their proper Fermi surfaces and some isoenergy surfaces (38) shifted from the Fermi surfaces by the *bare* Larmor frequency $2\beta H$. As a result, the generalized Landau theory of transverse dynamics somewhat resembles the theory of binary Fermi liquids which has been used so far[1] for a description of the longitudinal spin dynamics exclusively.

In the limits of low density or polarization, the equations in different transverse quasi-particles coincide with each other, and reduce to a single Silin-Leggett equation of the standard Landau theory.

In a general case, the molecular field, i.e. the Landau interaction, is given by a 4-component integral interaction operator (36). The components of this operator are expressed via the exact irreducible vertex part with the help of the integral equation (23), and cannot be represented, as it is usually assumed, by some limit of the full vertex.

In a homogeneous setting, both types of transverse quasi-particles precess around the magnetic field with the same bare Larmor frequency. However, the spin-up and spin-down quasi-particles are dressed in a

different way which leads to a noticeable dephasing of precession in inhomogeneous situations when the driving force includes different components of the interaction operator for different quasi-particles. This dephasing leads to an inhomogeneous broadening which manifests itself effectively as some strong zero-temperature relaxation associated with imaginary off-shell contributions.

Technically, this zero-temperature attenuation arises from complex terms in mass operators away from Fermi surfaces for pure spin states, temporal non-locality resulting in complex shifts of energy components of participating 4-momenta, and from momentum derivatives of the vertex functions in off-shell directions.

The zero-temperature relaxation is related both to complex values of energies for transverse quasi-particles' states, and to a strong attenuation of spin waves of the Landau–Silin type even in the limit of large wavelengths.

The results of this paper are applicable not only to spin-polarized Fermi liquids, but to any binary or two-level Fermi liquid (or even to a Fermi liquid of multi-level particles, see Ref. 23). The nature of internal levels of constituting particles is absolutely irrelevant. What is important is the assumption that the energy separation between internal levels of *bare* particles should not depend on their momenta. Otherwise, all the results of this paper can be applied to any two-level Fermi liquids without noticeable modifications. Of course, an experimental observation of the discussed effects is simpler at low temperatures for spin systems than for two-level systems of any other nature because of readily accessible NMR or EPR techniques.

The next parts of this series will deal with low density Fermi liquids and with polarization expansions. We will present more detailed analytical and numerical results for the dynamic characteristics and the zero-temperature relaxation.

APPENDIX A

The full self-energy operator $\Sigma(P_1, P_2)$ can be expanded in skeleton diagrams containing thick lines (exact Green's functions) and bare vertices only. None of these diagrams can be reduced into two by cutting any two thick lines. Since bare vertices do not depend on the Green's functions, the variational derivative $(-i\delta\Sigma(P_1, P_2))/(i\delta G(P_3, P_4))$ is obtained by cutting one line in each of these skeleton diagrams in all possible ways, and ascribing the sets of 4-momenta and spin variables P_3 and P_4 to the new free ends. The sets of variables for the ends of the initial diagrams are P_1 and P_2. We

consider only small inhomogeneities. Therefore, one can represent the above 4-momenta in the following way:

$$P_1 = P + \frac{K}{2}, \qquad P_2 = P - \frac{K}{2}, \qquad P_3 = P' - \frac{K}{2}, \qquad P_4 = P' + \frac{K}{2}$$

where K is small.

The four-end diagrams, resulting from the cuts of $\Sigma(P_1, P_2)$ through one of the thick lines, are skeleton diagrams for some vertex part. We will prove now that the set of all these resulting skeleton diagrams corresponds to the full *irreducible* vertex part.

Suppose that it is possible to obtain a reducible diagram for the vertex by cutting one line in some self-energy skeleton diagram. We can present this reducible vertex diagram as it is shown in Fig. 1(a). Then the original self-energy diagram is obtained by connecting the two ends with variable sets P_3 and P_4 by a thick line (Fig. 1(b)). But the resulting self-energy diagram is not a skeleton diagram of the type described above: it can be
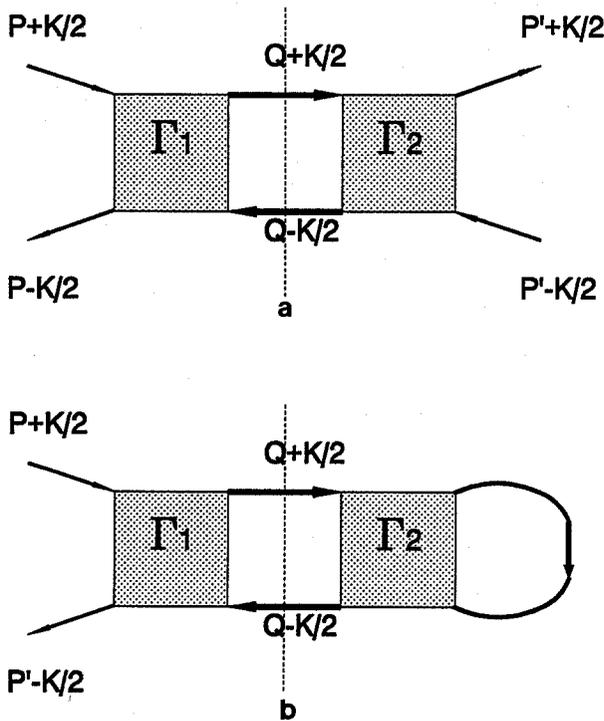
Fig. 1. (a) A reducible vertex diagram. (Appendix A). (b) The corresponding self-energy diagram (Appendix A).
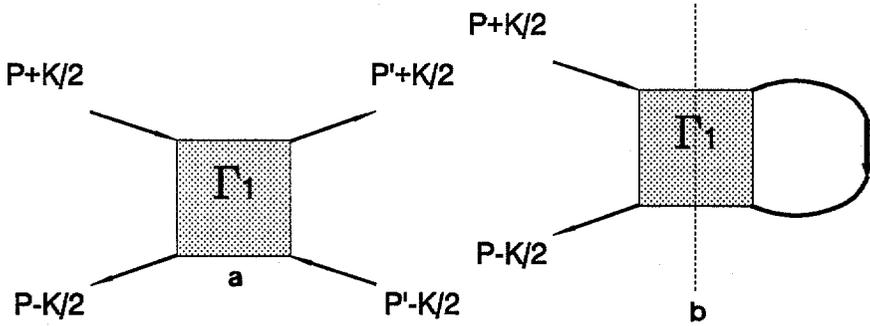
Fig. 2. (a) An irreducible vertex diagram. (Appendix A). (b) The corresponding self-energy diagram (Appendix A).

cut into two diagrams by the dashed line. Therefore, all the diagrams contributing to the variational derivative $-i\delta\Sigma(P_1, P_2)/i\delta G(P_3, P_4)$ belong to the set of diagrams constituting the irreducible vertex.

Now let us show that *all* of the diagrams of the irreducible vertex belong to this variational derivative. Indeed, any irreducible diagram $i\Gamma_1(P_1, P_3; P_2, P_4)$ (Fig. 2(a)) can be obtained from a self-energy diagram as shown on Fig. 2(b). Suppose this self-energy diagram is not a skeleton diagram of the above type. Then it can be cut into two by cutting two lines. Both of these lines should be in $i\Gamma_1(P_1, P_3; P_2, P_4)$, since a vertex part cannot be cut into two by cutting one line. But this means that $i\Gamma_1(P_1, P_3; P_2, P_4)$ can be cut into two by cutting two lines going in the opposite directions with almost equal momenta, which contradicts to the assumption that the original vertex part is irreducible. Thus we have proved that the variational derivative contains all and only all the diagrams entering the full irreducible vertex.

It is also obvious that each of the irreducible vertex diagrams is generated by only one skeleton diagram for self-energy (Fig. 2(b)), that is the diagram obtained by connecting the two ends with variable sets P_3 and P_4 by a thick line. Therefore

$$\frac{\delta\Sigma(P_1, P_2)}{\delta G(P_3, P_4)} = -i\tilde{\Gamma}(P_1, P_3; P_2, P_4)$$

where $\tilde{\Gamma}(P_1, P_3; P_2, P_4)$ is the full irreducible vertex part.

APPENDIX B

We want to prove that the first derivative of the irreducible vertex over k is zero at $k = 0$. In principle, one can always achieve this by a proper choice of variables. Our choice of incoming $(P + K/2, P' - K/2)$ and

P+K/2, ↑   P-K/2, ↓   P+K/2, ↑   P-K/2, ↓

$$\Gamma$$

$$\Gamma$$

P'-K/2, ↓ a   P'+K/2, ↑   P'-K/2, ↓ b   P'+K/2, ↑
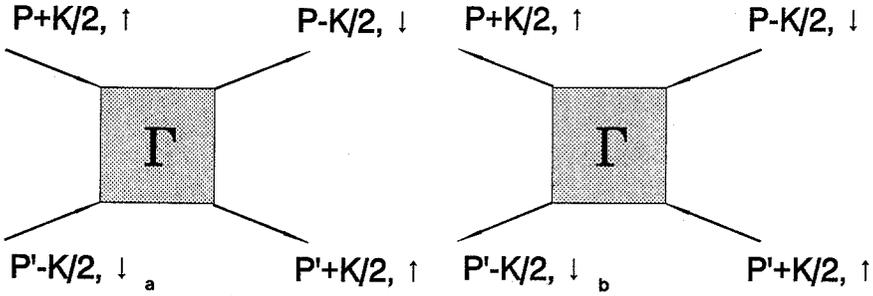
Fig. 3. (a) The full irreducible vertex. (Appendix B). (b) The diagram obtained from the full irreducible vertex by changing the directions of all the arrows on the lines (Appendix B).

outgoing $(P - K/2, P' + K/2)$ 4-momenta has been done in such a symmetric form partially in order to ensure the disappearance of this derivative.

Consider a diagram for the full irreducible vertex part (Fig. 3(a)). If we change the directions of all the arrows on the lines (not the spin states!), neither the Green's functions nor the bare vertices change. Therefore, the full irreducible vertex part does not change either (Fig. 3(b)). As a result,

$$\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P') = \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P', P) = \tilde{\Gamma}_{\downarrow\uparrow,\uparrow\downarrow}(-K; P, P')$$

On the other hand, the irreducible vertex should be invariant if we change all the spin arrows and the direction of the magnetic field to the opposite. Since $K_0 = 2\beta H + \alpha k^2$, K_0 changes to $-K_0$ under this transformation within the first order in k. This means that with the same accuracy

$$\tilde{\Gamma}_{\downarrow\uparrow,\uparrow\downarrow}(-K; P, P') = \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(\tilde{K}; P, P'), \quad \tilde{K} = (K_0, -\mathbf{k})$$

But this, in turn, means that the irreducible vertex is invariant under $\mathbf{k} \to -\mathbf{k}$ transformation, and

$$\left.\frac{\partial \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')}{\partial \mathbf{k}}\right|_{k=0} \equiv 0$$

APPENDIX C

First we substitute Eq. (22) into Eq. (7) and cancel out $G_{\downarrow\downarrow} - G_{\uparrow\uparrow}$ from both sides:

$$\left(\delta\omega - \frac{\mathbf{pk}}{m} - \Sigma_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)\right)\tilde{g}(P)$$

$$= -i \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')\left[G_{\downarrow\downarrow}\left(P' + \frac{K}{2}\right) - G_{\uparrow\uparrow}\left(P' - \frac{K}{2}\right)\right]\tilde{g}(P')\frac{d^4P'}{(2\pi)^4} \quad \text{(C.1)}$$

Representing the difference of the Green's functions $G_{\downarrow\downarrow} - G_{\uparrow\uparrow}$ in Eq. (C1)

by its singular and regular parts via Eq. (20) and regrouping terms, we obtain

$$\int \left[\delta^{(4)}(P - P') - \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')\phi_r\left(P' + \frac{K}{2}, P' - \frac{K}{2}\right)\right]$$

$$\times \left(\delta\omega - \frac{\mathbf{p}'\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}\left(P' + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P' - \frac{K}{2}\right)\right)\tilde{g}(P')\frac{d^4P'}{(2\pi)^4}$$

$$= \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P')\phi_s\left(P' + \frac{K}{2}, P' - \frac{K}{2}\right)\tilde{g}(P')\frac{d^4P'}{(2\pi)^4} \qquad (C.2)$$

On the other hand, we can rewrite Eq. (23) as

$$\tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, P') = \int \left[\delta^{(4)}(P - Q) - \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, Q)\phi_r\left(Q + \frac{K}{2}, Q - \frac{K}{2}\right)\right]$$

$$\times \mathfrak{C}(K; Q, P')\frac{d^4Q}{(2\pi)^4} \qquad (C3)$$

Now we can combine Eq. (C.3) and Eq. (C.2) into a single equation:

$$\int \left[\delta^{(4)}(P - Q) - \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(K; P, Q)\phi_r\left(Q + \frac{K}{2}, Q - \frac{K}{2}\right)\right]$$

$$\times \left[\left(\delta\omega - \Sigma_{\downarrow\downarrow}\left(Q + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(Q - \frac{K}{2}\right)\right)\tilde{g}(Q)\right.$$

$$\left. - \int \mathfrak{C}(K; Q, P')\phi_s\left(P' + \frac{K}{2}, P' - \frac{K}{2}\right)\tilde{g}(P')\frac{d^4P'}{(2\pi)^4}\right]\frac{d^4Q}{(2\pi)^4} = 0 \qquad (C.4)$$

We assume that the integral equations (23) and (C.3) have a unique solution $\mathfrak{C}$. This means that the operator represented by the kernel of the integration over Q in Eqs. (C.3) and (C.4) is non-degenerate and, therefore, has an inverse. Consequently, the only solution of the integral equation (C.4) is given by

$$\left(\delta\omega - \Sigma_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \frac{K}{2}\right)\right)\tilde{g}(P)$$

$$- \int \mathfrak{C}(K; P, P')\phi_s\left(P' + \frac{K}{2}, P' - \frac{K}{2}\right)\tilde{g}(P')\frac{d^4P'}{(2\pi)^4} = 0$$

which is equivalent to Eq. (21).

In the Landau–Silin case an equation analogous to Eq. (23) (see e.g. Ref. 22):

$$\Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow}(P, P') = \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(P, P')$$

$$+ \int \tilde{\Gamma}_{\uparrow\downarrow,\downarrow\uparrow}(P, Q)\phi_r(Q, Q)\Gamma^\omega_{\uparrow\downarrow,\downarrow\uparrow}(Q, P')\frac{d^4Q}{(2\pi)^4} \qquad (C.5)$$

can be used as an alternative definition of the Landau function Γ^ω. The presence of the 4-vector K in Eq. (23), in contrast with Eq. (C.5), incorporates the temporal and spatial non-localities. These non-localities become essential for a highly polarized Fermi liquid. Then Eq. (21) plays the role of a non-local analog of the Landau-Silin equation. Note, that the temporal and spatial non-localities are taken into account not only in the generalized Landau function $\mathfrak{C}$, but also in the singular part $\phi_s(P+K/2, P-K/2)$.

APPENDIX D

The singularity in the full vertex in the particle-hole channel leads to the following well-known relation between the full and the irreducible vertices[22]:

$$\Gamma_{\gamma\delta,\alpha\beta}(K; P, P') = \tilde{\Gamma}_{\gamma\delta,\alpha\beta}(K; P, P')$$

$$-i \int \tilde{\Gamma}_{\gamma\xi,\alpha\eta}(K; P, Q) G_{\xi\zeta}\left(Q+\frac{K}{2}\right) G_{\eta\kappa}\left(Q-\frac{K}{2}\right) \Gamma_{\kappa\delta,\zeta\beta}(K; Q, P') \frac{d^4Q}{(2\pi)^4}$$

This singularity, being nested in the product of the Green's functions, originates from the pinching of the contour of integration over q_0 between the poles of the two Green's functions. The Green's functions can be written as:

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = \frac{Z_{\uparrow,\downarrow}(\mathbf{p})}{p_0 - \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) + \mu + i0 \operatorname{sign} p_0} + G^r_{\uparrow,\downarrow}(P)$$

The contour of integration over q_0 can be closed in an infinite semicircle in either the upper or the lower half-plane. Therefore, the poles of the Green's functions are important only if they appear in the different half-planes, so that one cannot avoid them by any choice of the contour of integration. The residue at the corresponding pole will give the singular part of the product of the Green's functions ϕ_s leading to Eq. (19).

The pinching of the contour between the poles of the Green's functions occurs when the 3-momentum arguments of the Green's functions appear on the different sides of their respective Fermi spheres: one inside its Fermi sphere and the other outside. Then one of the Green's functions corresponds to a particle and the other to a hole. The region of the $\mathbf{p}$-space where such a situation occurs is shown by a shaded area on Fig. 4(a) and Fig. 4(b) for a non-polarized and a polarized Fermi liquid respectively. The vector $\mathbf{k}$ describes the shift between the centers of the spheres for spin-ups and spin-downs.
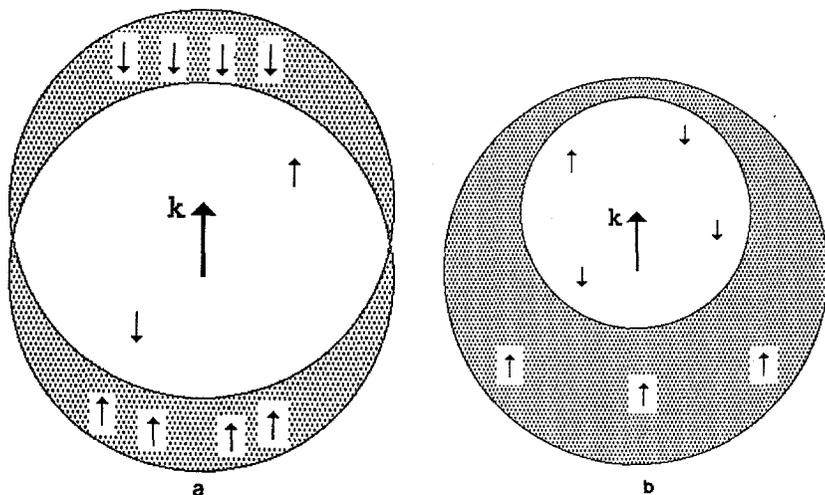
Fig. 4. (a) Two Fermi spheres for spin-up and spin-down particles in an unpolarized Fermi liquid. The shaded area is responsible for the particle-hole singularity. The spheres are shifted with respect to each other by a vector **k** (Appendix D). (b) Two Fermi spheres corresponding to spin-up and spin-down particles for a polarized Fermi liquid. (Appendix D).

In case of a non-polarized Fermi-liquid (Fig. 4(a)), the shaded area responsible for the singularity is defined by

$$\left| \mathbf{q} + \frac{\mathbf{k}}{2} \right| > p_F, \qquad \left| \mathbf{q} - \frac{\mathbf{k}}{2} \right| < p_F$$

or

$$\left| \mathbf{q} + \frac{\mathbf{k}}{2} \right| < p_F, \qquad \left| \mathbf{q} - \frac{\mathbf{k}}{2} \right| > p_F$$

Therefore, the singular part ϕ_s contains a factor $\theta(|\mathbf{q}+\mathbf{k}/2| - p_F) - \theta(|\mathbf{q}-\mathbf{k}/2| - p_F)$. At small **k** this factor reduces to $\mathbf{qk}\delta(q - p_F)$. Thus in the limit $\mathbf{k} \to 0$, when the centers of the spheres merge, the shaded area disappears (Fig. 5(a)) and the singularity in the particle-hole channel is suppressed.

In case of a finite polarization (fig. 4(b)) the shaded area is defined by:

$$\left| \mathbf{q} + \frac{\mathbf{k}}{2} \right| > p_\downarrow, \qquad \left| \mathbf{q} - \frac{\mathbf{k}}{2} \right| < p_\uparrow,$$

provided that $\mathbf{qk}/m < 2\beta H$. Therefore the singular part ϕ_s contains a similar factor $\theta(|\mathbf{q}+\mathbf{k}/2| - p_\downarrow) - \theta(|\mathbf{q}-\mathbf{k}/2| - p_\uparrow)$. Unlike the unpolarized case, the shaded area remains finite at any choice of **k** (Fig. 5(b)): since the radii of the spheres $p_\uparrow$ and $p_\downarrow$ are different, one cannot eliminate the shaded area
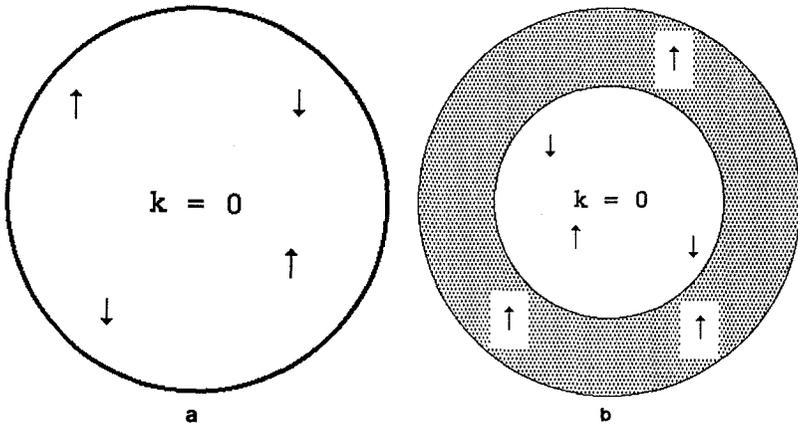
Fig. 5. (a) The shaded area disappears when the Fermi spheres for spin-ups and spin-downs merge in case of an unpolarized Fermi liquid with $k = 0$ (no shift) (Appendix D). (b) In case of a polarized Fermi liquid, the shaded area always remains finite, because the Fermi spheres for spin-ups and spin-downs cannot merge at any k, including $k = 0$ (Appendix D).

by any shift of the spheres. Thus the the above factor $\theta(|\mathbf{q}+\mathbf{k}/2| - p_\downarrow) - \theta(|\mathbf{q}-\mathbf{k}/2| - p_\uparrow)$ entering the singular part ϕ_s does not vanish.

APPENDIX E

A Green's function $G_{\uparrow\uparrow}$ or $G_{\downarrow\downarrow}$ of a Fermi liquid can be separated into its singular and incoherent parts in the following way (see Eq. (27)):

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = \frac{Z_{\uparrow,\downarrow}(\mathbf{p})}{p_0 - \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) + \mu + i0 \, \text{sign} \, p_0} + G_{\uparrow,\downarrow}^r(P) \tag{E.1}$$

We want to extract the δ-type singularity associated with the pole of the Green's function (E.1) with the help of the theory of generalized functions. Consider an integration of the Green's function (E.1) with some arbitrary function $f(p_0)$ over p_0 along the real axis $Im \, p_0 = 0$. Suppose the pole of the Green's function $p_0 = \varepsilon(\mathbf{p}) - \mu$ is in the upper half-plane of the variable p_0 (see Fig. 6). The integration over p_0 along the real axis is equivalent to the integration along the contour C_1 plus the integral along C_2. Contour C_1 is parallel to the real axis and circles the pole from below infinitely close to it. The closed contour C_2 consists of the real axis, the horizontal line $Im \, p_0 = Im \, \varepsilon(\mathbf{p}) - 0$ infintely close to C_1, and two infinitely remote vertical stretches. The integrals along these two vertical paths, obviously, vanish. The direction of integration along the upper horizontal line in C_2, $Im \, p_0 = Im \, \varepsilon(\mathbf{p}) - 0$, is *opposite* to that in C_1. Therefore, the sum of these integrals is zero.
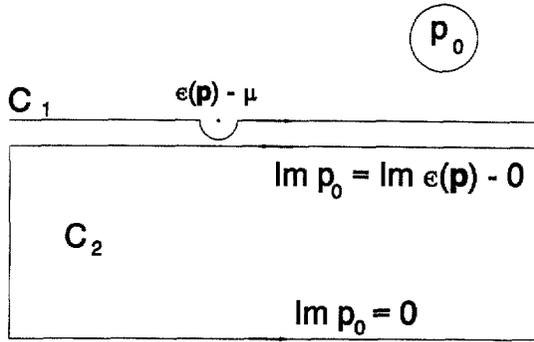
Fig. 6. The contours of integration C_1 and C_2 in the complex plane of variable p_0 (Appendix E).

The overall integration along C_2 is equal to the sum of residues inside this contour. It contributes only to the function G^r in (E.1), since the pole of the Green's function is left out of the contour. The integration along C_1 should be performed using a well-known equation,

$$\frac{1}{x-i0} = P\frac{1}{x} + i\pi\delta(x) \tag{E.2}$$

where P indicates that the integration should be taken as a principle value. Indeed, the integration along the semicircle around the pole gives a half-residue at the pole, while the integration along the rest of the contour C_1 gives the principle value of the integral and, therefore, contributes only to G^r. As a result, the singular part of the Green's function associated with its pole (E.1) can be presented by a δ-function:

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = i\pi Z_{\uparrow,\downarrow}(\mathbf{p})\delta(p_0 - \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) + \mu) + \tilde{G}^r_{\uparrow,\downarrow}(P) \tag{E.3}$$

where $\tilde{G}^r$ includes not only G^r (E.1), but also the regular terms coming from the principle value of the integral along the contour C_1, as well as the overall integration along the closed contour C_2.

If the pole of the Green's function appears in the lower half-plane, then the analogous formula is:

$$G_{\uparrow\uparrow,\downarrow\downarrow}(P) = -i\pi Z_{\uparrow,\downarrow}(\mathbf{p})\delta(p_0 - \varepsilon_{\uparrow,\downarrow}(\mathbf{p}) + \mu) + \tilde{G}^r_{\uparrow,\downarrow}(P) \tag{E.4}$$

Note, that the Green's function of an ideal gas can be decomposed into singular and regular parts in a similar way:

$$\frac{1}{p_0 - \dfrac{p^2}{2m} + \mu + i0\,\mathrm{sign}\,p_0} = P\frac{1}{p_0 - \dfrac{p^2}{2m} + \mu} - i\pi\delta\left(p_0 - \frac{p^2}{2m} + \mu\right)\mathrm{sign}\,p_0 \tag{E.5}$$

The product of the Green's functions contains the particle-hole singularity if the poles of the two Green's functions appear in the different half-planes (see e.g. Appendix D). Then this product can be rewritten as follows:

$$
G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right) G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right)
$$

$$
= i\pi \left[\theta\left(\varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) - \mu\right) - \theta\left(\varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) - \mu\right) \right]
$$

$$
\times \left[Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \delta\left(p_0 - \frac{\delta\omega}{2} - \beta H - \varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) + \mu\right) G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) \right.
$$

$$
\left. - Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) \delta\left(p_0 + \frac{\delta\omega}{2} + \beta H - \varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) + \mu\right) G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right) \right]
$$

$$
+ i\phi_r\left(P + \frac{K}{2}, P - \frac{K}{2}\right) \tag{E.6}
$$

where ϕ_r is trivially related to the functions $\tilde{G}^r$ above. The difference of θ-functions in (E.6) assures that the poles of the Green's functions are on the different sides of the contour of integration.

The Green's function can be also expressed through the mass operator:

$$
G_{\uparrow\uparrow,\downarrow\downarrow}(P) = \left[p_0 - \frac{p^2}{2m} \pm \beta H - \Sigma_{\uparrow\uparrow,\downarrow\downarrow}(P) - \mu\right]^{-1} \tag{E.7}
$$

Substituting Eq. (E.7) into Eq. (E.6) and simplifying it, we obtain:

$$
G_{\uparrow\uparrow}\left(P - \frac{K}{2}\right) G_{\downarrow\downarrow}\left(P + \frac{K}{2}\right) = i\pi\left[\theta\left(\varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) - \mu\right) - \mu\left(\varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) - \mu\right) \right]
$$

$$
\times \left[Z_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \delta\left(p_0 - \varepsilon_\uparrow\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) + \mu\right) \right.
$$

$$
\left. + Z_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) \delta\left(p_0 - \varepsilon_\downarrow\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) + \mu\right) \right]
$$

$$
\times \frac{1}{\delta\omega - \dfrac{\mathbf{p}\mathbf{k}}{m} - \Sigma_{\downarrow\downarrow}\left(P + \dfrac{K}{2}\right) + \Sigma_{\uparrow\uparrow}\left(P - \dfrac{K}{2}\right)}
$$

$$
+ i\phi_r\left(P + \frac{K}{2}, P - \frac{K}{2}\right) \tag{E.8}
$$

A simple comparison of Eq. (E.8) with Eq. (19) leads to Eq. (30) for ϕ_s:

$$
\phi_s\left(P+\frac{K}{2}, P-\frac{K}{2}\right) = \pi\left[\theta\left(\varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)-\mu\right)-\theta\left(\varepsilon_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)-\mu\right)\right]
$$

$$
\times\left[Z_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)\delta\left(p_0+\frac{\delta\omega}{2}+\beta H-\varepsilon_\downarrow\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)+\mu\right)\right.
$$

$$
\left. +\left[Z_\uparrow\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)\delta\left(p_0-\frac{\delta\omega}{2}-\beta H-\varepsilon_\uparrow\left(\mathbf{p}-\frac{k}{2}\right)+\mu\right)\right]\right]
$$

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