Pressure Diffusion and Sound Absorption in Spin-Polarized Quantum Systems

K. D. Ivanova* and A. E. Meyerovich

Institute for Physical Problems, 117334 Moscow, USSR

Pressure diffusion and related phenomena are studied in the cases of Fermi liquids and dilute gases with arbitrary degree of quantum degeneracy. An equation is derived expressing the (spin) pressure diffusion ratio through partial viscosities of (spin) components of systems. The exact values of corresponding transport coefficients are given for the cases of spin-polarized Boltzmann or degenerate quantum gases and spin-polarized Fermi liquids. The influence of surface slip effects on diffusion properties of spin-polarized quantum systems is discussed. The results may be used for gaseous and liquid ^3He, $^3He \uparrow$ $-^4He$ solutions, gases $H \uparrow$ and $D \uparrow$, and other spin-polarized or binary quantum systems.

1. INTRODUCTION

The most striking manifestations of macroscopic quantum effects in spin-polarized quantum gases and liquids are connected with polarization peculiarities of transport and other kinetic phenomena (for reviews see ref. 1). The polarization of spin systems leads not only to a considerable change of the usual transport coefficients (viscosity η, thermal conductivity $\varkappa$, and spin diffusion D_2), but also to the existence of additional transport coefficients [spin thermal diffusion, spin pressure diffusion, spin second (bulk) viscosity]. The reason is that if we are interested only in processes not accompanied by the rotation of magnetization (i.e., of the spin polarization vector), then a spin-polarized system is macroscopically formally equivalent to a mixture of several subsystems corresponding to particles (or quasiparticles) with different values of spin projection. In this case a polarized system of particles with spin S must be characterized by the same set of kinetic coefficients as a usual $(2S+1)$-component mixture of gases

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*Permanent address: Institute of Solid State Physics, Bulgarian Academy of Sciences, Sofia, Bulgaria.

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or liquids. The calculations of corresponding transport coefficients for spin-polarized systems may be performed using the well-developed methods and formulas for kinetic calculations in many-component mixtures (such an approach to transport phenomena in spin-polarized gases and Fermi liquids is somewhat analogous to the Wang Chang and Uhlenbeck approach to the kinetics of gases of molecules with complicated internal energy spectra).

Certainly the simplest and most realistic case corresponds to systems of spin-$\frac{1}{2}$ fermions, for which the polarization causes the system to become similar to some binary mixture (of spin-up and spin-down components). The diffusion properties of such a system are governed by an equation analogous to the diffusion equation[2] for binary mixtures, while the role of concentration of a dissolved component is played by the degree of spin polarization.[3] If the magnetic interaction of (quasi) particles is mostly of exchange origin (as in the case of ^{3}He systems), then the diffusion processes mainly conserve the number of particles in each of the spin components.

The diffusion of the magnetic moment (without a change of its direction!) in a spin-polarized liquid or gaseous system of spin-$\frac{1}{2}$ fermions can be characterized using the analogy with usual binary mixtures by three diffusion coefficients: the spin diffusion coefficient D_s, the spin thermal diffusion coefficient D_{sT}, and the spin pressure diffusion coefficient D_{sp}. In this case the spin diffusion current is proportional to the gradients of the spin polarization α (the analog of component concentration in binary mixtures), pressure P, and temperature:

$$\mathbf{j}_s = -D_2 N[\nabla \alpha + (K_{sp}/P)\nabla P + (K_{sT}/T)\,\nabla T] \tag{1}$$

where we introduce dimensionless spin pressure and thermal diffusion ratios $K_{sp} \equiv PD_{sp}/D_sN$ and $K_{sT} \equiv TD_{sT}/D_sN$ (N is the number of fermions per unit volume).

Usually the pressure diffusion ratio is calculated according to simple thermodynamic relations[2]; in our case this means that

$$K_{sp} = \frac{P}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\frac{\mu_+}{m_+} - \frac{\mu_-}{m_-}\right)\right]^{-1}_{P,T}; \qquad \rho = \frac{N}{2}[m_+(1+\alpha) + m_-(1-\alpha)] \tag{2}$$

where $\mu_\pm$ and $m_\pm$ are the chemical potentials and (effective) masses of (quasi) particles with spin projections $\pm\frac{1}{2}$. Corresponding results for spin-polarized quantum gases and Fermi liquids have been obtained.[3] However, it has been known for a long time[4] that a viscous flow of a liquid or gaseous mixture leads to a considerable renormalization of the pressure diffusion coefficient (2). Such a renormalization is caused by the necessity of taking into account the second spatial derivatives of the overall mass velocity of

a liquid or a gas in the expression for a diffusion current: according to the Navier–Stokes equation, the second derivatives of the total mass velocity **u** are proportional to the first spatial derivatives of pressure; in a stationary viscous flow $\Delta \mathbf{u} = (1/\eta)\nabla P$.

In our case such a renormalization is extremely important,[1] especially for a nondegenerate (Boltzmann) spin-polarized quantum gas, since in this gas the value of pressure diffusion ratio (2) without viscous renormalization is equal to zero, reflecting the absence of pressure diffusion in a mixture of ideal Boltzmann gases of particles with equal masses (see below).

In this paper we present the results of calculations of (spin) pressure diffusion coefficients for different spin-polarized systems of spin-$\frac{1}{2}$ fermions, including the viscous renormalization. In the next section we derive a simple expression for K_{sp} generalizing the results of ref. 4 so to incorporate binary systems with arbitrary degrees of quantum degeneracy of distribution functions. This expression is used for evaluation of K_{sp} in dilute spin-polarized degenerate and Boltzmann gases (Section 3) and in polarized Fermi liquids (Section 4). Since the general expression for K_{sp} obtained in Section 2 relates the pressure diffusion ratio K_{sp} to the partial viscosities $\tau_{\pm}$ of the spin components, one must determine the values of $\eta_{\pm}$ and corresponding relaxation times $\tau_{\pm}$ in order to evaluate K_{sp}. These values of $\eta_{\pm}$ and $\tau_{\pm}$ are important not only for the pressure diffusion, but also for the problems connected with sound absorption at not very low frequencies and with other high-frequency phenomena.

The pressure diffusion coefficient calculated in Sections 2–4 determines the local relation between the density of diffusion (spin) current and the pressure gradient in the volume of bulk systems. However, it is well known that slip effects at the walls lead to the appearance of large diffusion flows in the boundary layers of a gas in the presence of temperature or pressure gradients. Therefore, if one is interested in the relation between the total diffusion flow in some tube and the difference of pressures at the ends of this tube, one must take into account not only the bulk pressure diffusion currents, but also the surface contribution of the Knudsen boundary layer with the thickness of the order of the (quasi) particle mean free path (Section 5).

We are interested mainly in low-temperature ^{3}He $\uparrow$ -^{4}He solutions (where, depending on ^{3}He concentration, the ^{3}He impurity system may represent either a dilute gas of fermions with an arbitrary degree of quantum degeneracy or a comparatively dense Fermi liquid) and in a pure, normal ^{3}He $\uparrow$ Fermi liquid. Some of the results may be used also for gases of atomic H $\uparrow$ or D $\uparrow$ if, as in usual experimental conditions, only two hyperfine components must be taken into account. Certainly the results may be used for other binary mixtures of gases or Fermi liquids.

2. PRESSURE DIFFUSION IN BINARY MIXTURES

The value of the pressure diffusion coefficient for the stationary viscous flow of a binary mixture of classic (Boltzmann) ideal gases was calculated in ref. 4. It is necessary to generalize the results of ref. 4 in two directions. First, for quantum systems one must have an expression for K_{sp} valid at different degrees of quantum degeneracy, including completely degenerate Fermi gases. Moreover, since the description of Fermi liquids is based on the gaseous representation for the quasiparticle system, the corresponding expression for K_{sp} must include also the case of a degenerate, dense Fermi liquid. Second, the derivation of expressions for K_{sp} in ref. 4 was to a large extent based on a 13-moments Grad expansion. This is a serious limitation of the results, for the Grad method is certainly not the only one for calculating the transport coefficients even for a mixture of Boltzmann gases. For this reason it may be useful to derive some general expressions for K_{sp} without making concrete the method of solving the kinetic equation.

Let us begin with the case of a dilute spin-polarized gas of spin-$\frac{1}{2}$ (quasi) particles with quadratic energy spectrum $\varepsilon(\mathbf{p})$ at an arbitrary degree of the quantum degeneracy of the distribution functions for up and down spins. The kinetic equation for the diagonal-in-spin components $n_{\pm}$ of the single-particle density matrix $\hat{n}$ has the form

$$\frac{\partial n_{\pm}}{\partial t} + \frac{\partial \varepsilon_{\pm}}{\partial \mathbf{p}} \frac{\partial n_{\pm}}{\partial \mathbf{r}} - \frac{\partial \varepsilon_{\pm}}{\partial \mathbf{r}} \frac{\partial n_{\pm}}{\partial \mathbf{p}} = St_{\pm}(n_+, n_-) \tag{3}$$

where the indices $(+)$ and $(-)$ determine the spin components of the system. Although in a dilute gas the (effective) mass of (quasi) particles does not depend on the sign of the spin projection, for the sake of consequent generalizations to the cases of Fermi liquid or binary mixtures of gases of particles with different masses it may be convenient to introduce formally different (effective) masses $m_{\pm}$ for particles with up and down spins and to use the equality $m_+ = m_-$ only in the final expressions for pressure diffusion in very dilute spin-polarized gases.

The kinetic equations (2) can be transformed in a standard manner following, for example, refs. 4 and 5. The mean (mass) velocity of the particles is

$$\mathbf{u} = (m_+ N_+ + m_- N_-)^{-1} \int (m_+ n_+ \mathbf{v}_+ + m_- n_- \mathbf{v}_-) \, d\Gamma, \qquad d\Gamma = d^3p/(2\pi\hbar)^3 \tag{4}$$

where

$$N_{\pm} = \int n_{\pm} \, d\Gamma, \qquad N_+ + N_- = N$$

are the numbers of (quasi) particles with up and down spins per unit volume. Multiplying Eqs. (2) by $m_{\pm}(\mathbf{v}_{\pm}-\mathbf{u})$ and integrating over momenta, one can easily obtain the equation

$$\frac{d\mathbf{J}_{\pm}}{dt}-\mathbf{J}_{\pm}\frac{\partial u}{\partial r}+\frac{5}{3}\frac{\mathbf{J}_{\pm}\partial \mathbf{u}}{\partial \mathbf{r}}+\hat{\varepsilon}\mathbf{J}_{\pm}+\frac{d\mathbf{u}}{dt}+\frac{\partial P_{\pm}}{\partial \mathbf{r}}+\frac{\partial \Pi_{\pm}}{\partial \mathbf{r}}=\mathscr{G}_{\pm} \tag{5}$$

with the notations

$$\mathbf{J}_{\pm}=m_{\pm}\int n_{\pm}(\mathbf{v}_{\pm}-\mathbf{u})\, d\Gamma$$

$$P_{\pm}=\tfrac{1}{3}m_{\pm}\int (\mathbf{v}_{\pm}-\mathbf{u})^2 n_{\pm}\, d\Gamma$$

$$\hat{\varepsilon}\equiv \varepsilon_{ik}=\partial u_i/\partial r_k+\partial u_k/\partial r_i-\tfrac{2}{3}\delta_{ik}\,\partial u_l/\partial r_l$$

$$\hat{\Pi}_{\pm}\equiv \Pi_{\pm ik}=m_{\pm}\int n_{\pm}[(\mathbf{v}_{\pm}-\mathbf{u})_i(\mathbf{v}_{\pm}-\mathbf{u})_k-\delta_{ik}(\mathbf{v}_{\pm}-\mathbf{u})^2/3]\, d\Gamma$$

$$\mathscr{G}_{\pm}=m_{\pm}\int d\Gamma\, (\mathbf{v}_{\pm}-\mathbf{u})St_{\pm}$$

$$d/dt\equiv \partial/\partial t+\mathbf{u}\,\partial/\partial \mathbf{r}$$

After linearization and the usual simplifying assumptions including the momentum conservation law

$$(m_+N_++m_-N_-)\, d\mathbf{u}/dt+\partial P/\partial \mathbf{r}+\partial \hat{\Pi}/\partial \mathbf{r}=0 \tag{6}$$

Eq. (4) reduces to

$$\frac{\partial P_{\pm}}{\partial \mathbf{r}}-\frac{m_{\pm}N_{\pm}}{m_+N_++m_-N_-}\left(\frac{\partial P}{\partial \mathbf{r}}+\frac{\partial \hat{\Pi}}{\partial \mathbf{r}}\right)+\frac{\partial \Pi_{\pm}}{\partial \mathbf{r}}=-\mathscr{G}_{\pm} \tag{7}$$

where the pressure is $P=P_++P_-$ and the viscous stress tensor has the form $\hat{\Pi}=\hat{\Pi}_++\hat{\Pi}_-$. For isotropic viscous systems one can always introduce "partial (spin) viscosities" $\eta_{\pm}$ so that

$$\hat{\Pi}_{\pm}=-\eta_{\pm}\hat{\varepsilon} \tag{8}$$

while the total viscosity is $\eta\equiv \eta_++\eta_-$. In the case of a stationary viscous motion it is possible to neglect the term with $d\mathbf{u}/dt$ in (6),

$$\partial P/\partial \mathbf{r}+\partial \hat{\Pi}/\partial \mathbf{r}=0 \tag{9}$$

As a result, Eq. (7) with the help of Eqs. (8) and (9) take the very simple form

$$\partial P_{\pm}/\partial \mathbf{r}-(\eta_{\pm}/\eta)\,\partial P/\partial \mathbf{r}=-\mathscr{G}_{\pm} \tag{10}$$

We are not interested in thermodiffusion, and can regard the temperature as constant, $T = $ const. Then direct calculations show the validity of the following identity for the quantities $P_\pm$ of (5):

$$dP_\pm = N_\pm \, d\mu_\pm$$

In the usual approach to diffusion problems (without viscous renormalization) the diffusion coefficients and diffusion currents in binary gas mixtures are determined [instead of Eqs. (10)] from the kinetic equation

$$N_\pm \, \partial\mu_\pm/\partial r - [m_\pm N_\pm/(m_+ N_+ + m_- N_-)] \, \partial P/\partial r = -\mathcal{G}_\pm \qquad (11)$$

[the quantities $\mathcal{G}_\pm$ still have the form (5)] with an additional condition $\mathbf{j}_+ + \mathbf{j}_- = 0$ necessary due to the degeneracy of the system of two equations (11). Comparison of Eq. (10) and (11) together with the identity for $dP_\pm$ immediately leads to the possibility of representing the diffusion current at $T = $ const in the form

$$\mathbf{j}_s \equiv \mathbf{j}_+ - \mathbf{j}_- = \text{const} \times \left[\frac{\partial}{\partial r}\left(\frac{\mu_+}{m_+} - \frac{\mu_-}{m_-} \right) - \left(\frac{\eta_+}{m_+ N_+} - \frac{\eta_-}{m_- N_-} \right) \frac{1}{\eta} \frac{\partial}{\partial r} P \right]$$

Comparing this with Eq. (1) brings one the final expression for the (spin) pressure diffusion ratio:

$$K_{sp} = P \left[\frac{1}{m_+}\left(\frac{\partial\mu_+}{\partial P} \right)_{\alpha,T} - \frac{1}{m_-}\left(\frac{\partial\mu_-}{\partial P} \right)_{\alpha,T} - \frac{\eta_+/m_+ N_+ - \eta_-/m_- N_-}{\eta} \right]$$
$$\times \left[\frac{1}{m_+}\left(\frac{\partial\mu_+}{\partial\alpha} \right)_{P,T} - \frac{1}{m_-}\left(\frac{\partial\mu_-}{\partial\alpha} \right)_{P,T} \right] \qquad (12)$$

where $\alpha \equiv (N_+ - N_-)/N$ is the degree of spin polarization (or the concentration of a binary mixture).

Equation (12) means that for dilute gases with an arbitrary degree of quantum degeneracy the evaluation of the (spin) pressure diffusion ratio reduces to the calculation of partial viscosities and thermodynamic derivatives of chemical potentials for different (spin) components of a gas. In different cases one can use different adequate methods of calculating $\eta_\pm$ (e.g., Chapman-Enskog method, Grad expansion, etc.) depending on the peculiarities of the problem in question. The accuracy of calculations using Eq. (12) increases with decreasing temperature due to the existence of exact methods of solving the kinetic equation and evaluating kinetic coefficients for degenerate gases in contrast to Boltzmann ones.

The derivation of the expression for the (spin) pressure diffusion ratio in a polarized dense Fermi liquid does not differ considerably from the above one for gases. In this case the starting kinetic equation still has the form (3) and after not very cumbersome calculations it is possible to

reproduce formally Eqs. (10) and (12). Note that the Fermi liquid function in a dense Fermi liquid enters the expression for K_{sp} in (12) explicitly only through the thermodynamic derivatives of chemical potentials.

3. PRESSURE DIFFUSION AND SOUND ABSORPTION IN POLARIZED QUANTUM GASES. DILUTE ^{3}He $\uparrow$ –^{4}He SOLUTIONS

There are several very well-developed procedures for the evaluation of kinetic coefficients in different types of dilute gases. Therefore the calculation of the (spin) pressure diffusion ratio (12) may present technical difficulties rather than fundamental ones for dilute gases: one has only to apply some appropriate well-established scheme of calculation of partial viscosities for binary gas mixtures and obtain the values of $\eta_\pm$.

In our particular cases of spin-polarized quantum systems there are dilute systems of special interest, the so-called "quantum gases"[1]: gases of (quasi) particles with de Broglie wavelength Λ comparable to the atomic dimension a_0 or even larger than a_0. One of the most important types of such a gas is a gas of ^{3}He impurity quasiparticles in liquid ^{3}He-^{4}He solutions at low temperatures and low ^{3}He concentrations.[1] One can experimentally investigate this gas of ^{3}He impurities practically at all degrees of spin polarization and quantum degeneracy from the Boltzmann region $T \gg T_0$ down to the completely degenerate state $T \ll T_0$ [$T_0 = \hbar^2(3\pi^2 N)^{2/3}/2M$ is the Fermi temperature, N is the number of ^{3}He atoms per unit volume of the solution, and M is the ^{3}He quasiparticle effective mass; at saturated vapor pressure $M \approx 2.3m_3$ and $T_0 \approx 2.6x^{2/3}$ K, where m_3 is the ^{3}He atomic mass and x stands for the ^{3}He concentration in the solution]. For such a system it is rather easy to get into the quantum region $\Lambda \gtrsim a_0$ in both degenerate $T \ll T_0$ and Boltzmann (and also in intermediate) cases. The condition $\Lambda \gtrsim a_0$ corresponds to the inequalities

$$T, T_0 \lesssim \hbar^2/2Ma_0$$

with the quantity on the right-hand side of the order of 1 K. Therefore the system of ^{3}He quasiparticles in ^{3}He-^{4}He liquid solutions represents an example of a quantum gas, $\Lambda \gtrsim a_0$, at temperatures lower than 0.3 K and ^{3}He concentrations below 1%. Note that in spin-polarized dilute ^{3}He $\uparrow$ –^{4}He solutions the effective masses of quasiparticles with up and down spins are equal to each other, $m_+ = m_- = M$, in the main order in ^{3}He concentration.

All kinetic calculations and results simplify significantly in the ultraquantum limit[1] $\Lambda \gg a_0$ when the (quasi) particle interaction reduces mainly to s-wave scattering with amplitude $(-a)$ independent of momenta (in this sense kinetic calculations for ultraquantum gases are analogous to

those in the hard spheres model). Moreover, in the case of a gas of fermions an additional simplification arises as a result of the quantum identity of (quasi) particles, meaning the effectiveness of only the collisions of particles from different spin components for the s-wave scattering.

The thermodynamic derivatives in Eq. (12) become trivial for ideal spin-polarized Boltzmann gases, $T \gg T_0$, and K_{sp} in (12) takes the form

$$K_{sp} = 2(N_+ N_- / \eta N)(\eta_- / N_- - \eta_+ / N_+) \tag{13}$$

while the value of K_{sp} in (12) at $m_+ = m_- = M$ is equal to zero. The partial viscosities $\eta_\pm$ in Eq. (13) may be determined by Chapman–Enskog or Grad expansions (see, e.g., refs. 4 and 5), using standard formulas for binary mixtures of classical gases. For example, if we are interested in quantum gases, $\Lambda \gg a_0$, then, according to the data of refs. 1, 3, 4, and 6,

$$\eta_\pm / \eta = (2N_\pm^2 + \tfrac{1}{2} N_+ N_-)/(2N_+^2 + 2N_-^2 + N_+ N_-)$$

$$K_{sp} = 3(N_+ N_- / N)(N_- - N_+)/(2N_+^2 + 2N_-^2 + N_+ N_-) \tag{14}$$

For degenerate, $T \ll T_0$, spin-polarized gases direct thermodynamic calculation of K_{sp} in (12) yields [cf. (13)]

$$K_{sp} = \frac{3}{10\eta} [\eta_-(1+\alpha)^{5/3} - \eta_+(1-\alpha)^{5/3}] \frac{(1+\alpha)^{5/3} + (1-\alpha)^{5/3}}{(1+\alpha)^{2/3} + (1-\alpha)^{2/3}} \tag{15}$$

with $\alpha \equiv (N_+ - N_-)/(N_+ + N_-)$. Note that the evaluation of thermodynamic derivatives in (12) for a degenerate, nearly ideal gas in the main order in the concentration almost always means the necessity of consequent kinetic calculations in the ultraquantum limit $\Lambda \gg a_0$, since the corrections in a_0/Λ have the same order of magnitude as already neglected thermodynamic contributions of the order $N^{1/3}$. For the Boltzmann gases treated above this is not so, and the smallness of concentration $Na_0^3 \ll 1$ and of the quantum parameter a_0/Λ represents two different expansion parameters [in the Boltzmann case $\Lambda \sim \hbar/(MT)^{1/2}$, while for degenerate gases $\Lambda \sim N^{-1/3}$]. The values of partial viscosities of a degenerate polarized quantum gas may be determined according to the calculations and results of refs. 1, 3, and 7. Straightforward calculations result in

$$\eta_+ = \frac{2^{5/3}(3\pi^2 N)^{5/3} \hbar^5}{3(2\pi)^3 a^2 M^2 T^2} \frac{\mathbb{C}(\lambda_+)}{d^3(1+d^3)^{5/3}(5-3d^2)}, \qquad \lambda_+ = 1 - 2d^2 + \frac{6d^4}{5}$$

$$\eta_- = \frac{2^{2/3}(3\pi^2 N)^{5/3} \hbar^5}{3(2\pi)^3 a^2 M^2 T^2} \frac{d^5 \mathbb{C}(\lambda_-)}{(1+d^3)^{5/3}}, \qquad \lambda_- = \frac{1}{5}, \quad d \equiv \left(\frac{N_-}{N_+}\right)^{1/3} \tag{16}$$

$$\mathbb{C}(\lambda) = \frac{1-\lambda}{4} \sum_{\nu=0}^{\infty} \frac{4\nu+3}{(\nu+1)(2\nu+1)[(\nu+1)(2\nu+1)-\lambda]}$$

$$\frac{\eta_+}{\eta} \equiv 1 - \frac{\eta_-}{\eta} = \frac{2\mathbb{C}(\lambda_+)}{2\mathbb{C}(\lambda_+) + (5d^8 - 3d^{10})\mathbb{C}(\lambda_-)}$$

Equations (15) and (16) give the value of the (spin) pressure diffusion coefficient for the viscous flow of a degenerate quantum gas. This result differs considerably from the result of purely thermodynamic calculations[1,3] using Eq. (2),

$$(3/20)(1-d^2)^{1/3}[(1+\alpha)^{5/3}+(1-\alpha)^{5/3}][(1+\alpha)^{2/3}-(1-\alpha)^{2/3}]$$

The transport calculations of refs. 8 and 9 make it possible to calculate the pressure diffusion ratio (12) also for the intermediate (between Boltzmann and degenerate) temperatures in quantum gases. It is also possible to obtain the pressure diffusion ratio for semidegenerate gases[1,10]—gases with degenerate spin-up component and nondegenerate spin-down component. However, in the latter case the value of K_{sp} is certainly small due to the smallness of $N_-/N \ll 1$ and $\eta_-/\eta_+ \approx \eta_-/\eta \ll 1$.

The partial viscosities $\eta_\pm$ have more than only an auxiliary meaning and are important for various phenomena distinct from pressure diffusion. For example, partial viscosities of spin components can be measured independently of each other in some high-frequency phenomena when the parameters $\omega\tau$ may be different for both components due to the difference in characteristic relaxation times $\tau_\pm$ for up and down spins. One such phenomenon is the absorption and frequency dispersion of sound at finite frequencies (it is well known[2] that at low frequencies the absorption of sound depends only on total viscosity η). The most characteristic example is given by ^{3}He-^{4}He solutions at low temperatures and ^{3}He concentrations when the first-sound waves are mainly the oscillations of superfluid He II. The ^{3}He quasiparticles also participate in these oscillations and their influence is insignificant, since the sound velocity dominates in the sound absorption at low temperatures. To some extent the first-sound wave in dilute ^{3}He-^{4}He solutions represents an oscillating external field for ^{3}He quasiparticles. The attenuation of these sound waves is rather small in the hydrodynamic ($\omega\tau \ll 1$) and high-frequency ($\omega\tau \gg 1$) limits due to the smallness of ^{3}He concentration (the latter limit in helium corresponds to so-called high-frequency first sound). The spin polarization of solutions leads[1,3] to the splitting of the first-sound absorption maximum at frequency $\omega\tau = 1$ into two maxima at frequencies $\omega\tau_+ = 1$ and $\omega\tau_- = 1$, while the expression for the sound absorption coefficient differs from its usual hydrodynamic value[2] by the substitution

$$\eta \to \eta_+/(1+\omega^2\tau_+^2)+\eta_-/(1+\omega^2\tau_-^2) \tag{17}$$

instead of the viscosity η. Evidently one of the maxima (on the frequency $\omega = 1/\tau_+$) moves to vanishingly low frequencies and increases in value, while the second one decrease with increasing polarization. In dilute degenerate solutions the relaxation times $\tau_\pm$ are related to the partial viscosities

$\eta_{\pm}$ in (16) as

$$\tau_{\pm} = 5M\eta_{\pm}/N_{\pm}p_{\pm}^2, \qquad p_{\pm} = \hbar(6\pi^2 N_{\pm})^{1/3} \qquad (18)$$

while in the Boltzmann region (14), $\tau_{\pm} = \eta_{\pm}/N_{\pm}T$. A nearly complete polarization $\alpha \to 1$ means for quantum gases (14), (16) that $\eta_+, \tau_+ \to \infty$, $\eta_- \to 0$, and τ_- remains finite, thus determining the asymptotics of sound absorption at $\alpha \to 1$.

4. PRESSURE DIFFUSION AND SOUND IN POLARIZED FERMI LIQUID. CONCENTRATED ^{3}He $\uparrow$ –^{4}He SOLUTIONS AND NORMAL PURE LIQUID ^{3}He $\uparrow$

The evaluation of K_{sp} still must be carried out using Eq. (12). However, the results turn out to be much more complicated in the case of dense, spin-polarized Fermi liquids like pure liquid ^{3}He $\uparrow$ or ^{3}He $\uparrow$ –^{4}He solutions with high ^{3}He concentration (about several percent) than for the above cases of dilute gases. For example, the thermodynamic derivatives in Eq. (12), determined in accordance with the thermodynamic formulas of refs. 1 and 3 for polarized Fermi liquids, take the form

$$\left(\frac{1}{m_+}\frac{\partial\mu_+}{\partial P} - \frac{1}{m_-}\frac{\partial\mu_-}{\partial P}\right)_\alpha$$

$$= \frac{1}{m_+m_-}\frac{N_+m_-p_-(A_+^{(0)}m_- - B_-^{(0)}m_+) - N_-m_+p_+(A_-^{(0)}m_+ - B_+^{(0)}m_-)}{N_+m_-p_-(A_+^{(0)}N_+ + B_-^{(0)}N_-) + N_-m_+p_+(A_-^{(0)}N_- + B_+^{(0)}N_+)} \qquad (19)$$

$$\left(\frac{1}{m_+}\frac{\partial\mu_+}{\partial\alpha} - \frac{1}{m_-}\frac{\partial\mu_-}{\partial\alpha}\right)_P$$

$$= \frac{\pi^2\hbar^3 N^2}{m_+m_-}\frac{(A_+^{(0)}A_-^{(0)} - B_+^{(0)}B_-^{(0)})(N_+m_+ + N_-m_-)}{N_+m_-p_-(A_+^{(0)}N_+ + B_-^{(0)}N_-) + N_-m_+p_+(A_-^{(0)}N_- + B_+^{(0)}N_+)}$$

where the harmonics of the Fermi liquid function $\hat{f}(\mathbf{p}, \hat{\sigma}; \mathbf{p}', \hat{\sigma}')$ for the general case of a Fermi liquid with purely exchange interactions

$$f(\mathbf{p}, \hat{\sigma}; \mathbf{p}', \hat{\sigma}') = \psi_1(\mathbf{p}, \mathbf{p}')\hat{I}\hat{I}' + \psi_2(\mathbf{p}, \mathbf{p}')\hat{\sigma}\hat{\sigma}' + e\{\psi_3(\mathbf{p}, \mathbf{p}')\hat{\sigma}\hat{I}'$$

$$+ \psi_3(\mathbf{p}', \mathbf{p})\hat{I}\hat{\sigma}'\} + \psi_4(\mathbf{p}, \mathbf{p}')(e\hat{\sigma})(e\hat{\sigma}')$$

are defined by the relations

$$\Psi_{iab}^{(n)} = (n + \tfrac{1}{2})\frac{p_b m_b}{2\pi^2\hbar^3}\int \psi_i(p_a, p_b')P_n(\cos\chi)d\cos\chi$$

$$A_\pm^{(n)} = 1 + \Psi_{1\pm\pm}^{(n)} + \Psi_{2\pm\pm}^{(n)} + 2\Psi_{3\pm\pm}^{(n)} + \Psi_{4\pm\pm}^{(n)}$$

$$B_\pm^{(n)} = \Psi_{1\pm\mp}^{(n)} - \Psi_{2\pm\mp}^{(n)} \pm \Psi_{3\pm\mp}^{(n)} \mp \Psi_{3\mp\pm}^{(n)} - \Psi_{4\pm\mp}^{(n)}$$

Above, $\hat{\sigma}$ are the Pauli matrices, $\hat{I} \equiv \delta_{\alpha\beta}$ is the unitary spin operator, $\mathbf{e}$ is the unit vector along the direction of polarization, the indices a and b take the values $(+)$ or $(-)$, P_n are Legendre polynomials, and $m_{\pm}$ and $p_{\pm}$ are respectively the effective masses and Fermi momenta for quasiparticles with up and down spins. The values of effective masses $m_+ \neq m_-$ are related to the masses of bare (quasi) particles by the first harmonics of the f function $A_{\pm}^{(1)}, B_{\pm}^{(1)}$.[1]

The values of partial viscosities $\eta_{\pm}$ for (spin) components of a polarized (or binary) Fermi liquid may be obtained using the methods developed in refs. 1, 3, and 11:

$$\eta_{\pm} = \frac{2}{15} \frac{\hbar^3 p_- p_+^4}{T^2 m_{\pm}^4} \frac{1}{Z_{\pm}} \left[S_{\pm} + 6 \left(\frac{m_{\pm}}{m_{\mp}} \right)^2 \frac{p_+ p_{\mp}^2}{p_{\pm}^3} \frac{YS}{Z_{\mp}} \right]$$

$$Z_{\pm} = \frac{p_+}{p_{\mp}} \left\langle \frac{W_{\pm}}{4 \cos(\theta/2)} + \left(\frac{m_{\mp}}{m_{\pm}} \right)^2 \frac{W'}{f_{\pm}} \right\rangle$$

$$S_{\pm} = \left(\frac{1}{2} \right) \sum_{\nu=1}^{\infty} \frac{(4\nu-1)}{\nu(2\nu-1)} \frac{\nu(2\nu-1)-1+\lambda_{\mp}}{\gamma_{\nu}}$$

$$Y = \left\langle \frac{W'}{f_-^3} \frac{p_-}{p_+} \sin^2\theta \sin^2\left(\frac{\varphi}{2}\right) \left[\cos\theta + \frac{1}{f_+ f_-} \sin^2\theta \sin^2\left(\frac{\varphi}{2}\right) \right] \right\rangle \qquad (20)$$

$$S = \frac{1}{4} \sum_{\nu=1}^{\infty} \frac{4\nu-1}{\nu(2\nu-1)} \frac{1}{\gamma_{\nu}}$$

$$\gamma_{\nu} = \left[\nu(2\nu-1) - 1 + \frac{\lambda_+ + \lambda_-}{2} \right]^2 + \frac{(\lambda_+ - \lambda_-)^2}{4} - \frac{36 Y^2}{Z_+ Z_-}$$

$$\lambda_{\pm} = 6 X_{\pm} \left(\frac{m_{\mp}}{m_{\pm}} \right)^2 \left(\frac{p_-}{p_{\pm}} \right)^2 Z_{\pm}^{-1}$$

$$f_{\pm} = \left[1 + \left(\frac{p_{\pm}}{p_{\mp}} \right)^2 + 2 \frac{p_{\pm}}{p_{\mp}} \cos\theta \right]^{1/2}$$

$$X_{\pm} = \left\langle \frac{\sin^2\theta \sin^2(\varphi/2)}{f_-^3} \left[W' \left(1 - \frac{\sin^2\theta \sin^2(\varphi/2)}{f_{\pm}^2} \right) \right. \right.$$
$$\left. \left. W_{\pm} \left(\frac{m_{\pm}}{m_{\mp}} \right)^2 \frac{p_+^3}{p_{\mp}^3} \frac{f_-^3 \sin^2\theta \cos^2(\varphi/2)}{2^{2/3}(1+\cos\theta)^{5/2}} \right] \right\rangle$$

where $\langle \cdots \rangle$ means averaging over the angles, $W_+(\theta, \varphi)$ is the phenomenological scattering probability of spin-up quasiparticles on the Fermi sphere of radius p_+, W_- is the probability of mutual scattering of spin-down quasiparticles, and W' is the probability of scattering for quasiparticles with different spin projections.

The partial viscosities $\eta_\pm$ in (20) also determine the dispersion and attenuation of first sound in spin-polarized concentrated ^{3}He ↑ -^{4}He solutions at high frequencies [Eq. (17)]. In dense Fermi liquids the relaxation times for different components $\tau_\pm$ are related to their partial viscosities by $\tau_\pm = 5m_\pm \eta_\pm / N_\pm p_\pm^2$ instead of Eq. (18).

5. BOUNDARY CONDITIONS AND TOTAL DIFFUSION CURRENTS IN POLARIZED SYSTEMS

Slip effects at boundary surfaces for liquids or gases are well-known causes[12] of peculiar surface heat flows and (for multicomponent systems) diffusion currents. Since the contributions of the surface heat and diffusion flows to the total heat and diffusion flows through a tube are of the same order in a ratio of a mean free path to a characteristic dimension as contributions of first Burnett terms for bulk flows, one must always take into account in calculating overall diffusion and heat flows both bulk and surface contributions. This also must be done for pressure diffusion problems.*

In our case $T = $ const the boundary condition on the mass velocity of a gas near some solid surface has the form[12]

$$\mathbf{u}_t(z = 0) = \alpha_{11} \, \partial \mathbf{u}_t / \partial z |_{z=0} + \alpha_{12} \, \boldsymbol{\nabla}_t \mu |_{z=0} \tag{21}$$

where $\boldsymbol{\nabla}\mu = (1/m_+) \boldsymbol{\nabla}\mu_+ - (1/m_-) \boldsymbol{\nabla}\mu_-$, the index t denotes the components of vectors along the surface determined by the equation $z = 0$, and the coefficients α_{ik} correspond to slip effects near the solid boundary surface [in the absence of slip the boundary condition has the simple form $\mathbf{u}_t(z = 0) = 0$]. The values of α_{ik} are of the order of

$$\alpha_{12} \sim l/c \tag{22}$$

where l is the mean free path of (quasi) particles and c is the sound velocity. The nonzero mass velocity near the surface (21) means the existence of specific surface diffusion currents[12]

$$j_t^{(0)} = \alpha_{21} \, \partial u_t / \partial t |_{z=0} + \alpha_{22} \, \boldsymbol{\nabla}_t \mu |_{z=0} \tag{23}$$

The problem of calculating the total (pressure) diffusion current through a tube or flow channel must be dealt with in the following manner. First one has to express the distribution of overall mass velocity over some cross section of a tube $\mathbf{u}(\mathbf{r})$ through the spatial gradients of pressure $\boldsymbol{\nabla}P$

*The necessity of taking into account simultaneously surface and bulk terms for pressure diffusion in mixtures of usual Boltzmann gases is already known (see, e.g., refs. 13 and references therein). The results below are based on those communicated to use by A. M. Bishaev, V. S. Galkin, and V. A. Rikov.

and chemical potential $(1/m_+) \nabla \mu_+ - (1/m_-) \nabla \mu_-$. This must be done by solving the Navier–Stokes equation with the boundary condition (21) without taking into account any additional Burnett corrections to the equation; all such corrections result in higher order terms in the mean free path l. Then, with the help of Eqs. (1) and (12) one calculates the distribution of the density of the bulk diffusion current. This distribution has to be integrated over the chosen cross section of the tube and added to the surface diffusion current (23) integrated over the perimeter of this cross section. As a result one obtains a linear dependence of the total diffusion current I_D through the cross section of the tube on the gradients of pressure and chemical potential or concentration (spin polarization). The corresponding coefficients include the coefficients of (spin) diffusion, viscosity, (spin) pressure diffusion, and the set of surface coefficients α_{ik} in (21)–(23). However, some of these coefficients are related to each other by the Onsager principle of the symmetry of kinetic coefficients.

Let us illustrate the above procedure on the simplest example. Consider two large volumes of gas at different pressures P_1 and P_2 ($P_1 - P_2 \ll P_{1,2}$) connected with a tube of radius R (x axis). The temperature is supposed to be constant throughout the system. The velocity of the gas is governed by a stationary Navier–Stokes equation with the boundary condition (21) and is distributed over the cross section of the tube according to the quadratic Poiseuille law:

$$u_x(r) = -\frac{1}{4\eta} (R^2 - r^2 - 2R\alpha_{11}) \frac{dP}{dx} + \alpha_{12} \frac{d\mu}{dx} \tag{24}$$

(the pressure and the chemical potential depend only on x, but not on r). The total diffusion current through this cross section is determined by the bulk current [see (1) and (12)], $\mathbf{j}_s$ and the surface diffusion current $\mathbf{j}_t^{(0)}$ [see (23)]:

$$I_D = \int_0^R 2\pi r\, dr\, j_s^{(r)} + 2\pi R j_t^{(0)} = 2\pi R \left(\frac{\alpha_{21}}{2\eta} R \frac{dP}{dx} + \alpha_{22} \frac{d\mu}{dx} \right)$$

$$- \left(\frac{\partial \mu}{\partial \alpha} \right)_{P,T}^{-1} ND_s \left(\frac{d\mu}{dx} - \frac{\eta_+/N_+ m_+ - \eta_-/N_- m_-}{\eta} \frac{dP}{dx} \right) \pi R^2 \tag{25}$$

The entropy production in this case is given by

$$\dot{S} = -\frac{1}{T} \left[I_D \frac{d\mu}{dx} + \int \Pi_{xr} \frac{du_x}{dr} 2\pi r\, dr + 2\pi R \eta \left(u_x \frac{du_x}{dr} \right)_{r=R} \right] \tag{26}$$

where the component of stress tensor $\Pi_{xr} = \eta\, du_x/dr$. It is convenient to substitute du_x/dr in Eq. (26) by dP/dx using Eq. (24):

$$\dot{S} = -\frac{1}{T} \left\{ I_D \frac{d\mu}{dx} + \frac{dP}{dx} \left[\frac{\pi}{\eta} \int r^2 \Pi_{xr}\, dr + \pi R^2 u_x(r = R) \right] \right\} \tag{27}$$

In order to use the Onsager principle,[12] let us choose as thermodynamic forces the quantities

$$X_1 = d\mu/dx, \qquad X_2 = dP/dx \qquad (28)$$

Then Eq. (27) dictates the following choice of thermodynamic velocities:

$$\dot{x}_1 = \frac{1}{T}I_D, \qquad \dot{x}_2 = \frac{\pi}{\eta T}\int r^2\Pi_{xr}\,dr + \frac{\pi}{T}u_x(r=R)R^2 \qquad (29)$$

so that the entropy production (27) is equal to the sum $\dot{x}_iX_i$. The coefficients γ_{ik} in the linear relations $\dot{x}_i = \gamma_{ik}X_k$ are determined by the expressions for Π_{xr}, I_D [Eq. (25)], and u_x [Eq. (24)]. According to the Onsager principle, $\gamma_{12} = \gamma_{21}$, implying the equality

$$\alpha_{21} = \eta\alpha_{12} - ND_s(\partial\mu/\partial\alpha)_{P,T}^{-1}(\eta_+/N_+m_+ - \eta_-/N_-m_-) \qquad (30)$$

Equation (30) does not depend on the geometry of a flow channel. The above results must be generalized for the case of nonconstant temperature. In this case the boundary conditions must be supplemented by the slip term

$$\alpha_{13}\,\mathbf{\nabla}_t T|_{z=0}$$

in Eq. (21), an additional surface diffusion current

$$\alpha_{23}\,\mathbf{\nabla}_t T|_{z=0}$$

in Eq. (23), and by the equation for surface heat flow

$$\mathbf{q}_t^{(0)} = \alpha_{31}\,\partial\mathbf{u}_t/\partial z|_{z=0} + \alpha_{32}\,\mathbf{\nabla}_t\mu|_{z=0} + \alpha_{33}\,\mathbf{\nabla}_t T|_{z=0}$$

One must also take into account the expression for the bulk heat flow $\mathbf{q}$ in binary mixtures[2] and corresponding expressions for the entropy production. Then it is easy to reproduce the scheme (24)–(30) and to obtain Onsager relations of the type (30) for all nondiagonal coefficients of the 3×3 matrices $\gamma_{ik} = \gamma_{ki}$ and α_{ik}.

Generally in flow problems in polarized systems there are three possible flows (mass flow, spin diffusion current, and heat flow) caused by three driving forces (temperature, pressure, and polarization gradients). The 3×3 matrix of transport coefficients connecting total flows with thermodynamic forces involves both bulk and surface coefficients. By supplementing the information on transport coefficients from refs. 1 with the above calculations of the pressure diffusion coefficient, it is possible to evaluate all bulk contributions for spin-polarized gases and Fermi liquids. Unfortunately, there are no calculations of surface slip coefficients for polarized quantum gases and liquids. In principle, the surface slip for polarized quantum systems may be estimated in the same manner as for nonpolarized ones.[14] This estimate is simplified by the nearly specular character of surface

scattering for (quasi) particles in a low-temperature quantum system. Moreover, the surface scattering in a Boltzmann quantum gas does not depend on the (quasi) particle spins; such a dependence is exhibited only as a result of possible quantum degeneracy reflecting an effect of difference in velocities for up and down spins in polarized degenerate systems.

The above results for spin pressure diffusion may be important in different experimental situations (e.g., refs. 15–17). For example, the experimental conditions of ref. 17 correspond to the known temperature gradient, the known (zero) mass current, and to a small spin diffusion current determined mainly by depolarization processes in the low-temperature cell. Under these conditions (especially due to a very large temperature gradient) one may observe considerable pressure (density) and magnetization gradients between high- and low-temperature cells, resulting in a necessary renormalization of the parameters of the low-temperature cell.

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